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DIGITAL COMPUTATION STUDY, DYNAMIC VS KINEMATIC EQUATIONS

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ABSTRACT

At present, the simulation of a vehicle's motion is accomplished by use of either dynamic or kinematic equations. This report presents a method of simulating vehicular motion which is a compromise between these two methods. The equations developed here, although very close to kinematic equations in form, contain terms which represent the dynamics of the motions being simulated.

Expressions are developed to simulate the motion of submarine, surface vessel, and aircraft. Those for the aircraft describe its motion as a rigid body in space responding to the maneuvering orders; those for the submarine and surface vessel duplicate the accuracy and flexibility now attainable with simple dynamic differential equations. This latter is accomplished by using considerably less computer time and with less restriction on the size of the iteration interval.

FOREWORD

The simulation of vehicular motion has been the subject of many investigations. These investigations have been mainly concerned with the development of mathematical models for pilot training and for simple target motion. The models used in these two applications are presently well defined and widely used.

The model which is used in pilot training requires strict adherence to Newton's laws. In this situation there is direct control of the vehicle by the person receiving the training. Therefore the simulated environment must accurately describe the actual environment; the response of the simulated vehicle must closely approximate the response of the operational vehicle. This necessitates full dynamic simulation of the vehicle. On the opposite extreme of the simulation scale is the simulation of simple target motion. In this situation the control of the vehicle is completely divorced from the trainee. He is aware of the vehicle's motion by watching its path across a simulated radar screen, for example, but is not concerned with the forces causing the motion. Simple kinetic models are sufficient for this. There is a wide gap between these two degrees of vehicle simulation. The difference between them is demonstrated not only in the required accuracy and in their response characteristics, but also in their respective costs in time, money and in their computer requirements.

The cost of a model whose requirement is more responsive than the kinetic model, but vastly less than the full dynamic simulation is unreasonably high when the dynamic model with all its complexities must be utilized. Therefore this investigation was undertaken to define the area between the two extremes. Several models were developed which reflected various levels of simulation within this area. The development is based on kinematic descriptions of the motion, i.e. descriptions which consider knowledge of the type of maneuver being performed.

Four models are developed, two for which the control of the vehicle is directly (operator control) or indirectly (command control) under the control of the trainee and two for which the control is divorced from the trainee (instructor and program control). In general, the models differ with type of input, complexity of model and accuracy of response.

Models are developed for surface ships, submarines and aircraft. The

characteristics of the specific vehicle being simulated is reflected by the model so that, for example, the simulation of two submarines will not necessarily exhibit the same response. Due to the very complex nature of aircraft motion descriptions, the models for the aircraft are not developed as fully as those for the other types of vehicles.

It is anticipated that the results of this investigation, viz the mathematical models described in this report, will be of value in determining the least costly mathematical model with respect to computer requirements which will satisfy the requirements of simulating vehicles of varying accuracy and response characteristics.

The results obtained from programming the equations on a digital computer are discussed and compared in the report. They are graphically presented in a classified supplement which is available on special request. However, the contents of this supplement are not necessary in understanding or in utilizing the mathematical models.

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SECTION I

INTRODUCTION

1.1 CURRENT SIMULATION MODELS

At present, two kinds of mathematical models are used for the simulation of vehicular motion. They are the Dynamic Simulation Model and the Kinematic Simulation Model.

1.1.1 The Dynamic Simulation Model

The dynamic simulation model consists of three sets of equations. The first set consists of the dynamic differential equations $\underline{F} = m\underline{a}$ and $\underline{L} = I\underline{\alpha}$, where $\underline{F}$, $\underline{a}$, $\underline{L}$, and $\underline{\alpha}$ are vector quantities and I is a diagonal matrix. These are written as a set of six simultaneous differential equations for the six accelerations in the six degrees of freedom. Each of the six forces or torques is a function of one or more of the six velocities and of certain forcing functions such as engine thrust or rudder deflection angle. The constants of the six functions vary from one vehicle to the next, and sometimes from one environment to the next.

The second set of equations are the difference equations which update each velocity (at given intervals of time) as a function of past values of that velocity and its corresponding acceleration. These equations constitute a numerical solution technique of the type normally used to solve differential equations on a digital computer. Their accuracy is sensitive to the frequencies of the differential equations and to the interval between consecutive evaluations of the velocities. The equations in this second set are really kinematic equations, since they relate velocity to acceleration without taking into account the origin of the acceleration.

The third set of equations are kinematic, expressing displacement in the six degrees of freedom as functions of the six velocities.

This type of simulation model is called dynamic because the inputs to the model as a whole are the parameters of the forcing functions of the differential equations. The output is the motion of the vehicle. A model which yields motion as a result of applied forces is called dynamic.

1.1.2 The Kinematic Simulation Model

In a kinematic simulation model, the first set of equations used in the dynamic simulation model is not used. The second set is often retained, but in a simplified form. The third set of equations, which relates displacement to velocity, is the only one which is employed intact. This means that acceleration and sometimes even velocity must be supplied to the kinematic model from somewhere else in the simulation system. Unless dynamic equations are used to do this, the form of the input accelerations will be very simple.

The input generally takes one of three forms: acceleration as a linear function of time, constant acceleration, or an abrupt change in velocity. All three forms must be monitored and cut-off times supplies.

1.2 THE GAP LEFT BY CURRENT MODELS

1.2.1 Shortcomings of the Kinematic Model

It is clear from the form of the kinematic model that the accuracy of a kinematic simulation depends entirely on the accuracy of the acceleration or velocity used as input. At present, however, there is no systematic way of constructing this input. Current usage takes into account neither the parameters of the maneuver being executed nor the response characteristics of the vehicle simulated. Time delays between command and execution of a maneuver are approximated grossly or not at all. As a result, there is little difference between the simulation of a maneuver executed by one vehicle and the simulation of that same maneuver executed by any other vehicle. None are really wrong, but none are quite right.

It remains for some method to be evolved to monitor simply and accurately the inputs to the kinematic simulation model. This must be done in such a way as to reflect both the maneuver being executed and the vehicle being simulated.

1.2.2 Shortcomings of the Dynamic Model

The dynamic simulation model has none of the shortcomings of the kinematic model. The output from the dynamic model has all the characteristics necessary for a simulation faithful to the motion of the vehicle being simulated. In the dynamic model, however, the entire set of differential equations and accompanying difference equations must be evaluated at every iteration interval for the most simple as well as for the most complex maneuvers.

It is possible to simplify the set of differential equations described in Section 1.1.1 to a set that is only half as large. However, unless the simulator is to be used for training pilots for the vehicle, this is still too large. For many simulation purposes, the simulator nee. be able to duplicate only three or four different types of maneuvers.

Consider one of these: the turn. The kinematic model does not differentiate between the transition phases of different turns for the same ship or the same turn for different ships. The dynamic model does not take advantage of the fact that, for a given ship, one turn differs from the next only by its engine speed and rudder angle.

The same is true for the other types of maneuvers.

Therefore, the dynamic model must be modified. This modification must retain the fidelity of the original dynamic model while taking advantage of the extensive duplication of a very few types of maneuvers inherent in the operation of most simulation systems.

SECTION II

STATEMENT OF THE PROBLEM

There are two aspects to the problem:

a. To what extent is it possible to develop a third category of vehicle simulation models falling between the two that already exist, with respect to both fidelity and computer requirements?

b. In what simulation situations would it be advantageous to use models in this category?

SECTION III

METHOD OF PROCEDURE

3.1 PRELIMINARY CONSIDERATIONS

3.1.1 Dynamic Equations

An indication of what must be done was stated in the last paragraphs of Sections 1.2.1 and 1.2.2. Either the dynamic differential equations must be simplified or the simple kinematic equation must be modified. The simplification of a full set of dynamic differential equations can be a very difficult task. Fortunately, sets of simple dynamic equations are available for both the surface vessel and submarine. Both of these are sets of three simultaneous differential equations in three variables.

For the submarine, the time derivatives of speed, turn rate, and dive rate are presented as functions of the state of the system, engine speed, rudder deflection angle, and stern plane deflection angle. These three variables sufficiently describe the submarine's position and direction of motion at any time.

For the surface vessel, the time derivatives of speed, side-slip angle, and rate of change of heading are functions of the present state of the system, engine speed, and rudder deflection angle. These three variables completely describe the surface vessel's motion.

Both sets of dynamic equations give some indication of the vehicle's orientation, but the emphasis is on the motion of the vehicle as a moving point.

The important aspect of both these dynamic models, however, was that the differential equations could be solved analytically with a minimum of simplifying assumptions. The fact that the solutions were to be used for a limited number of maneuvers justified most of the simplifying assumptions.

The aircraft presented a more difficult task. The only differential equations available were very complex equations in six degrees of freedom. As a result, the model developed here had to be built up in an ad hoc manner. An attempt was made to fit equations to available response curves. The resulting model is incompletely developed, compared to the models for the submarine and surface vessel. The concepts of degrees of simplification and control situations, discussed in the next few sections, were not used. The construction of the model will be described in section 3.3.3.

3.1.2 Control Situations

Certain statements in section 1.2 discussed improvements necessary to the existing catalogue of vehicle simulation models. We implied that there are uses for simulation models for which the existing models are not adequate. If this is the case, then these uses

must be recognized before the improvements are developed. The following discussion is an attempt to establish a correspondence between types of uses and types of simulation models.

One way of categorizing simulation models is in terms of the agency that controls the motion of the simulated vehicle. In general, different control situations will call for different degrees of accuracy and fidelity of output. In certain cases, different control situations will demand different input parameters. No correspondence between control situations, input, and output can be definitive, but the one developed here will serve as a good indication of the lines along which the improved simulation models are to be constructed.

Four different control situations are used. The first two place the controlling agency on the vehicle itself, at the helm or on the bridge. They are called, respectively, the operator control situation and the command control situation. The last two place the controlling agency outside the vehicle. Control either rests in the hands of the instructor or is built into the program itself. These are the instructor control situation and the program control situation.

From the brief descriptions just given, it is possible to make some preliminary comments about the model that will correspond to each control situation. The model for the operator control situation must have as input those quantities normally under control of the helmsman. These are engine speed, rudder deflection angle and (where applicable) stern plane deflection angle. The output from this model should have no discontinuities, because the feedback to the helmsmen must be realistic.

The model used for the command control situation must be able to take input in either the same form as the operator controlled model or in the form of maneuver commands. The output need not have the same fidelity of response as that of the model used for the operator control situation.

In the instructor and program control situations, input is in the form of maneuver commands. The vehicles being simulated in these situations will generally be target vehicles. Fidelity requirements for target vehicles are not as great as for the other two situations; the emphasis here is on speed of computation. Both instructor and program, especially the latter, may be called upon to maneuver a large number of vehicles at once. The speed of computation must therefore be close to that of present kinematic models. Greater speed will be required of the program-controlled models than of the instructor-controlled models.

3.2 APPROACH

3.2.1 Four New Models

The first step in constructing simple simulation models from the available dynamic differential equations is to solve these equations and to present the solutions in the form of difference equations. In general, this procedure will give rise to the following situation.

Differential equation $\dot{v} = f($ ... .p., v.p.)

Difference equations v_n - ... $_{n-1} = hf_1(v_{n-1}$, m.p., v.p.)

where m.p. and v.p. stand for maneuver parameters and vehicle parameters, respectively. $v_n = v(t_n)$ and $v_{n-1} = v(t_{n-1})$ where $t_n - t_{n-1} = h$ for all n.

Sometimes f will be the same as f_1; sometimes it will be different. If it is different, this is because knowledge of the expected behavior of the maneuver parameters, together with certain restrictions on this behavior, has made possible the use of simplifying assumptions as to the nature of f. By varying the restrictions on the maneuver parameters, we arrive at the first two of the new simulation models.

In the first of these models, m.p. is allowed to vary in certain limited ways; in the second it is kept fixed. This exhausts the possibilities for simplifying the difference equation while it is in this form. The next step is to make $v_n - v_{n-1}$ independent of v_{n-1}.

When $v_n - v_{n-1}$ is independent of v_{n-1}, the difference equation has the form

$v_n - v_{n-1} = hf_3$(m.p., v.p.)

The advantage of this form is that, since m.p. is kept fixed, v_n is incremented by a constant amount at each iteration. Since v_{n-1} has been removed from the right side of the equation, however, v_n must be tested against some precomputed $_o v$ (ordered v) at each iteration. This requires less computation time than the inclusion of v_{n-1} in the increment function.

The next step is to let v_n jump from its original value to $_o v$ after a certain precomputed time delay has elapsed.

$v_n = v_o$ for $nh < \tau$

$v_n = {}_o v$ for $nh \geq \tau$

where $\tau = f_4$ (m.p., v.p.) and v_o is the value of v when t = 0.

f_3 and f_4 are found in the following way:

When the differential equation is solved, there results an equation of the form v = F(t, m.p., v.p.). Expressions for f_3 and f_4 are found by solving the following two equations:

$$\lim_{x \to \infty} \left[- \int_0^x F(t)\, dt + \int_0^{({}_o v - v_o)/f_3} (v_o + f_3 t)\, dt + \int_{({}_o v - v_o)/f_3}^{x} {}_o v \, dt \right] = 0 \qquad (3\text{-}1)$$

$$\lim_{x \to \infty} \left[-\int_0^x F(t)\,dt + \int_0^{f_4} v_o\,dt + \int_{f_4}^{x} {}_ov\,dt \right] = 0 \tag{3-2}$$

As described above, the four models give velocity at each iteration. Position as well as velocity is required as output from the complete model. Position is provided by the simple integration formula:

$$x_n = x_{n-1} + \frac{h}{2}(v_n + v_{n-1}). \tag{3-3}$$

Although this formula is very simple, errors in x_n do not feed back into the formulas for v_n.

This formula for x_n amounts to an approximation of the curves for v by straight-line segments between the points v_n. By examining Figures 3-1 through 3-4 in the next section it can be seen that, for each of Cases 1 through 4, such an approximation is still more accurate than the next, less accurate, case. Furthermore, all we really know about v is the location of the points v_n, so there is no loss in accuracy from using Equation 3-3 for x_n.

3.2.2 A Name for Each New Model

Section 3.1.2 describes four control situations and the nature of the input and output for each one. This description, not intended to be exhaustive, did however provide enough information to set up a correspondence between the four control situations and the four types of simulation models described in section 3.2.1. The correspondence is not unique or definitive, but rather serves as a guide for the use and evaluation of the models.

Output curves (Figure 3-1 through 3-4) are presented for $\dot{C}$. Curves for v have the same shape and can be obtained by substituting ${}_ov$ for ${}_o\dot{C}$ and v_o for $\dot{C}_o$. Curves for dive have certain differences which are described in section 3.3.1.3.

3.2.2.1 Operator Control

The equation for the first model is of the form

$$v_n - v_{n-1} = hf_1(v_{n-1}, \text{m.p., v.p.}) \tag{3-4}$$

where the m.p. are not fixed. Here, $v_n - v_{n-1}$ is proportional to ${}_ov - v_{n-1}$, where ${}_ov$ is a function of m.p. The variation that is allowed in m.p. is limited; its value is computed as a function of those quantities under the control of the helmsman. These quantities are allowed to vary in the way they ordinarily would in the execution of a single straightforward maneuver.

developed. This model has the properties required in a model for the operator control situation. Therefore, Case 1 will be known as the operator control model.

The output from the operator control model will resemble Figure 3-1.

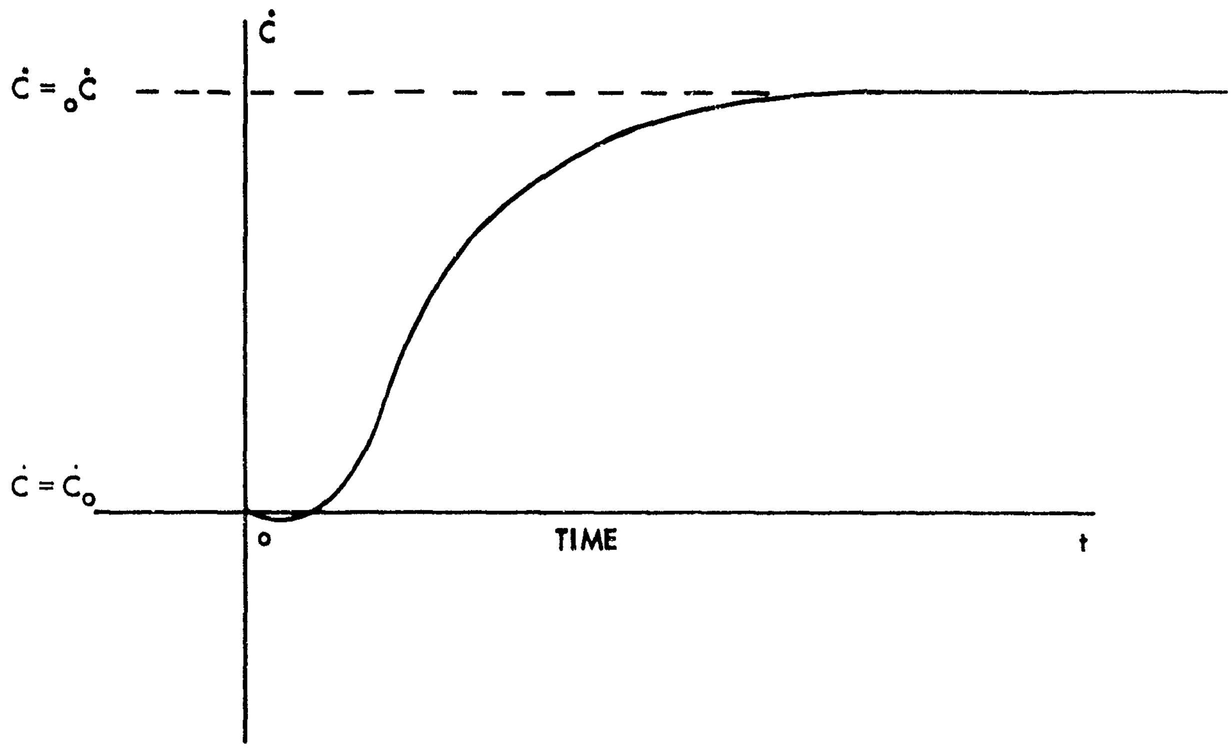

Figure 3-1. Output from Operator Control Equation for $\dot{C}$

3.2.2.2 Command Control

The equation for the second model is of the form

$$v_n - v_{n-1} = hf_2(v_{n-1}, \text{m.p.}, \text{v.p.}) \tag{3-5}$$

where m.p. is fixed. $v_n - v_{n-1}$ turns out to be proportional to ${}_o v - v_{n-1}$, where ${}_o v$ is a function of m.p. To make up for the variation that usually occurs in m.p. when a new maneuver is started, a time delay is used before allowing v_n to change. The value of m.p., however, is derived from the maneuver command.

Because m.p. is not allowed to vary, the output from this model is slightly less accurate than the output from Case 1, the operator control model. This model satisfies one of the two possible sets of requirements for the command control situation. The other is ordinarily satisfied by the operator control model, although certain limited variations are sometimes allowed in the m.p. in Case 2. Case 2 will therefore be known as the command control model.

The output from the command control model will resemble Figure 3-2.

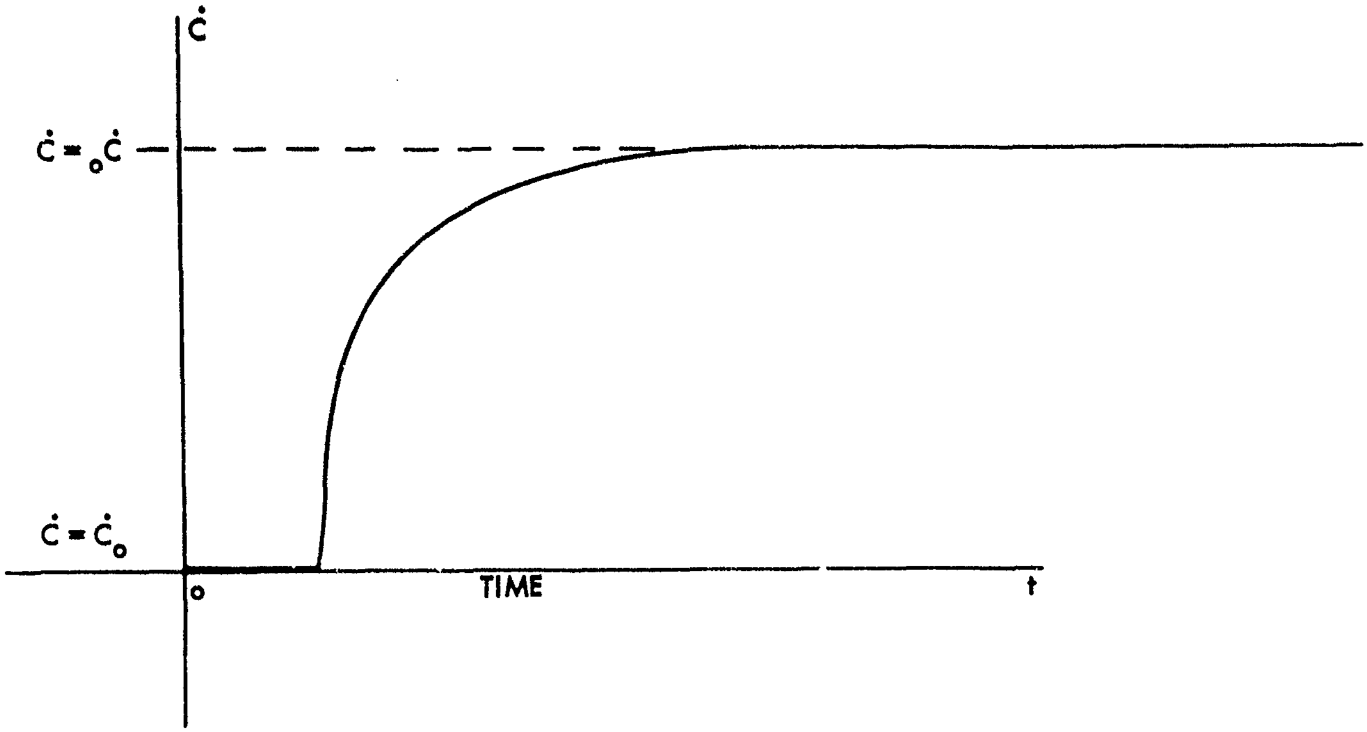

Figure 3-2. Output from Command Control Equation for $\dot{C}$

3.2.2.3 Instructor Control

The equation for the third model is of the form

$$v_n - v_{n-1} = hf_3(\text{m.p.}, \text{ v.p.}) \tag{3-6}$$

where the m.p. is fixed. The v.p. is always fixed since it is impossible to change vehicles in the middle of a maneuver. v_n is monitored so that it is always bounded by v_o and ${}_ov$.

Since the values for m.p. are fixed, they must be a function of maneuver commands.

The removal of the v_{n-1} dependence changes the nature of the behavior of v_n. Instead of approaching ${}_ov$ asymptotically, v_n changes linearly with time until it equals ${}_ov$. This makes the output from this model less realistic than the output from Case 2. Speed of computation is much greater, however, because the increment in v_n need not be recomputed.

Thus Case 3 satisfies the requirements that we were able to establish for a model for the instructor control situation. It will therefore be called the instructor control model.

The output from the instructor control model will resemble Figure 3-3.

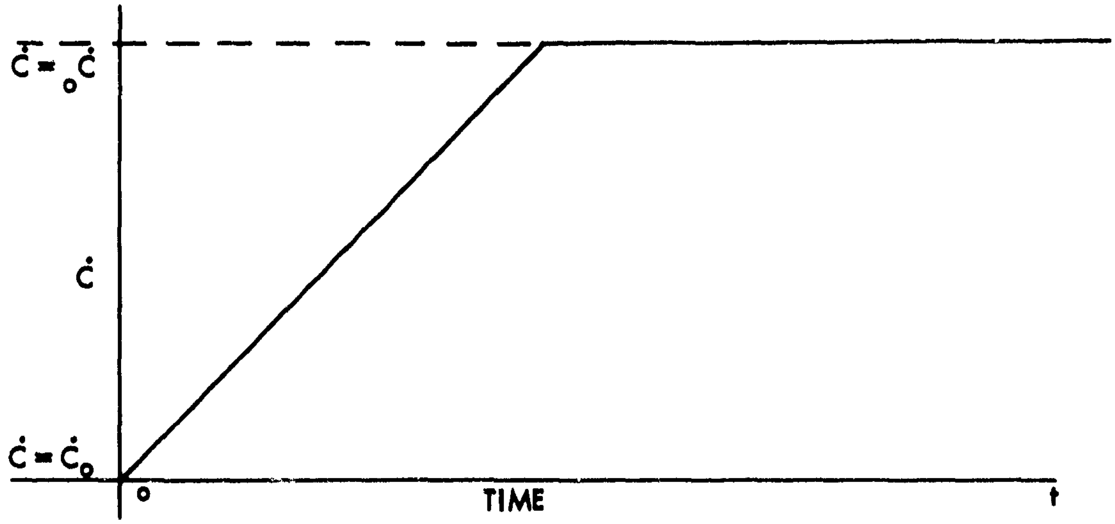

Figure 3-3. Output from Instructor Control Equation for $\dot{C}$

3.2.2.4 Program Control

The equation for the fourth model is of the form

$$v_n = v_0 \text{ for } nh < f_4 \text{ (m.p., v.p.)}$$
$$v_n = {}_0v \text{ for } nh \geq f_4 \text{ (m.p., v.p.)} \tag{3-7}$$

where the m.p. are fixed. v_0 is the value of v when t = 0 and ${}_0v$ is a function of m.p. and v.p. The m.p. values come from maneuver command inputs.

The output from this model is less accurate than the output from the instructor control model, for values of time less than 2 f_4. For larger values of time, the two outputs are usually equal. This model requires less computation time, however. It is easier to count time than to compare the value of v_n with ${}_0v$. Furthermore, it is easier to compute displacement from a fixed v than from a varying one.

Thus Case 4 satisfies the requirements for the program control situation, and the model will be known as the program control model. Its output is shown in Figure 3-4.

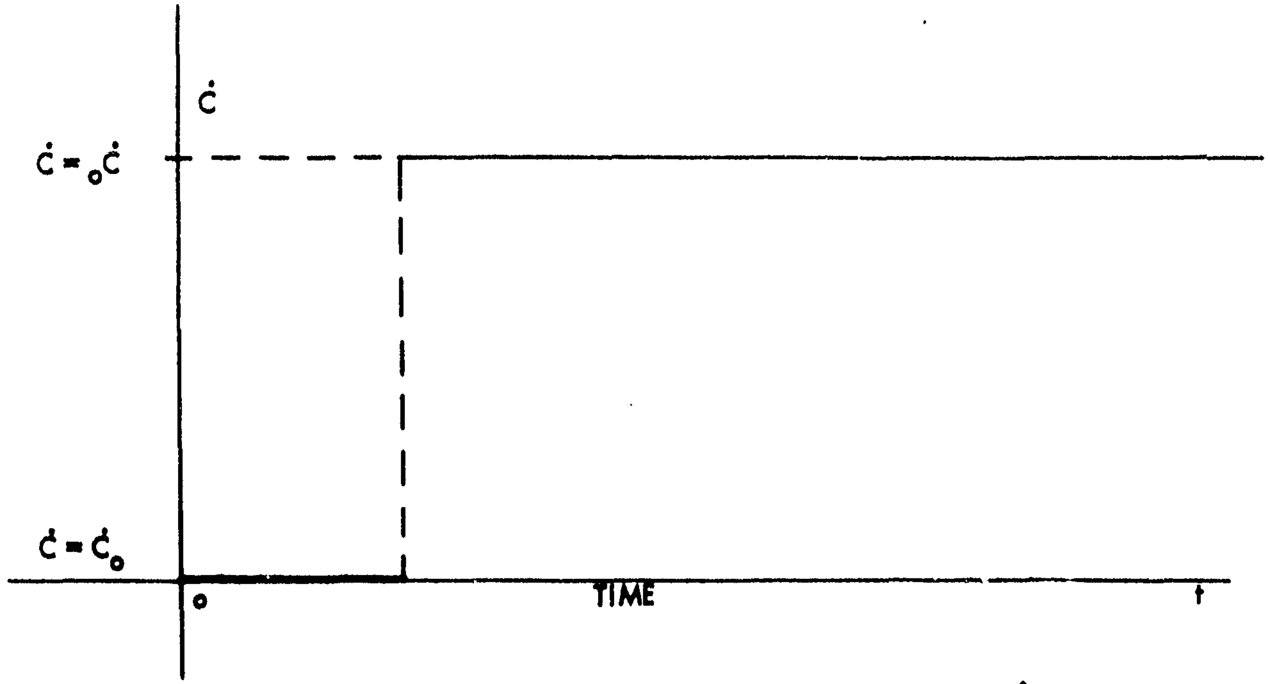

Figure 3-4. Output from Program Control Equation for $\dot{C}$

3.2.3 Kinematic Equations

Each of the four types of simulation models developed in this report has been given a name. (See section 3.2.2.) It remains for some sort of nomenclature to be associated with the group of four. Consider, once again, the form of the equations.

The first two represent simplifications of the dynamic differential equations, which were solved for a limited class of maneuvers. The solutions were put into the form of simple difference equations. These difference equations monitor acceleration as a function of the difference between current velocity and a fixed, pre-computed velocity.

The latter two equations are of the type usually contained in kinematic models, but with certain significant improvements. In the first of these two a constant acceleration is used, computed from the characteristics of the vehicle and the maneuver being simulated. The second of the two uses a discontinuous change in constant velocities. This change is delayed in such a way that the overall displacement is very close to that using the constant acceleration equation.

What we have, then, is a set of modified kinematic equations. Strictly speaking, a kinematic equation relates two or more kinematic variables such as acceleration, velocity and position. The models developed here are based on this type of equation, but with a significant addition. Each has built into it a means of monitoring acceleration in accordance with the maneuver being executed and the vehicle being simulated.

Every use of kinematic equations requires some algorithm to control the acceleration. The equations developed here are kinematic equations with the acceleration algorithms built into them. There is therefore ample justification for calling them kinematic equations.

3.3 MECHANICS OF SOLUTION

The methods used to derive equations for the command, instructor, and program control models from the equations for the operator control model are indicated in section 3.2.1. Those cases which required a departure from these methods will be described below. The main emphasis in this section, however, will be on the development of the operator control difference equation from the dynamic differential equations.

The complete analytic development of all the kinematic models is presented in Appendix A. Appendix B contains a full description of all the models, their equations, and the steps involved in their implementation on a digital computer.

3.3.1 Submarine

Reference 1 names the document that contains the differential equations for the motion of a submarine, equations which hold for all submarines. The properties of individual submarines are reflected in the values of the constants A_1 through A_{11}. These are constant for all the maneuverings of any one submarine, but differ from one submarine to the next. The exception to this is the surfaced submarine. When a submarine surfaces, not only do the values of the constants A_1 through A_{11} change, but probably the form of the equations as well. Therefore, use of the submarine kinematic equations developed here is restricted to submerged submarines.

The submarine dynamic equations are:

$$\dot{S} = A_1 \{{}_0S - (1 + A_2|\delta_r|)S\}\{{}_0S + (1 + A_2|\delta_r| + A_3)S\} \qquad (3\text{-}8)$$

$$\ddot{C} = -\{A_4S\dot{C} + A_5\dot{C}|\dot{C}| + A_6S^2\delta_r\} \qquad (3\text{-}9)$$

$$\ddot{D} = -\{A_7S\dot{D} + A_8\dot{D}|\dot{D}| + A_9D + A_{10}\dot{C}^2 + A_{11}S^2\delta_s\} \qquad (3\text{-}10)$$

The units are yards, degrees, and seconds. S is speed, C is course angle and D is dive angle. δ_r is rudder deflection angle, and δ_s is stern plane deflection angle. ${}_0S$ is the speed for which the engine is set.

3.3.1.1 Speed

The differential equation for speed was solved directly. The solution is an expression of the form $S = (a + be^{ct})/(d + fe^{ct})$. This leads to a difference equation of the form

$$S_n - S_{n-1} = f(S_{n-1}, e^{ch}).$$

For h sufficiently small, this can be expressed as h times a bilinear function of $_oS$ and S_{n-1}. The latter is used as the operator control equation.

3.3.1.2 Turn

A graph was constructed which depicts the differential equation for a submarine turn as a curve of $\ddot{C}$ versus $\dot{C}$. The portion of the curve representing the behavior of the submarine during an actual turn is very nearly a straight line. The equation for this straight line wa· constructed, keeping the intercepts intact as a function of speed and rudder angle.

If speed is kept constant and rudder angle constrained to be a linear function of time, then the equation for the straight line can be integrated directly for $\dot{C}$. This was done, and the current value of speed S_n substituted for the constant S in the solution. This leads to the desired difference equation for $\dot{C}_n$ of the form shown in equation 3-4.

This method of solution involves the quantity $\underline{\delta_r}$. The use of this quantity is a graphic example of the limitations of the kinematic models developed in this report. The kinematic equations for turn are to be used for a definite submarine turn with a fixed rudder angle. The rudder angle can vary while the submarine is moving into the turn or leveling out of the turn. However, unless the rudder is held fixed for the greater part of the turn or, at most, allowed to vary around some fixed value, the kinematic model will be inaccurate. That fixed value which the rudder deflection angle has for the greatest part of the turn is denoted by $\underline{\delta_r}$.

$\underline{\delta_r}$ appears in the difference equation for the operator control model. The operator control model therefore requires, as input, more than just the current value of the rudder angle. It must also have, as input, the rudder angle which the operator intends to use as the principal rudder angle of the turn maneuver.

3.3.1.3 Dive

The differential equation for submarine dive contains terms for dive angle, rate of change of dive angle, and rate of change of rate of change of dive angle. This leads to kinematic equations for submarine dive of a different form from all the other submarine and surface vessel kinematic equations.

The term containing $\dot{C}^2$ is ignored, since multiple maneuvers are not within the scope of the kinematic equations developed here.

The development of the four kinematic models for dive is sufficiently different from the corresponding development for the other submarine and surface vessel maneuvers to warrant a detailed explanation. In addition, the approach and results are different for each of the four cases.

a. Submarine Dive: Case 1

The curve of $\ddot{D}$ and $\dot{D}$ was drawn. As in the case of submarine turn, the curve is very nearly a straight line over the expected range of activity. The $\ddot{D}$ intercept, which is a function of D and δ_s, is preserved by the straight line approximation. The slope of the line is found by minimizing the integral of the square of the error over the expected range of activity. This leads to a linear second-order non-homogenous differential equation for D.

The forcing function is proportional to δ_s. The differential equation is solved assuming δ_s to be a linear function of time. Since the differential equation is second order, two starting values are needed for the difference equation. The difference equation is found from the solution to the differential equation by the following technique:

The solution to the differential equation is of the form:

$$D = K_1 e^{A_1 t} + K_2 e^{A_2 t} + f(\delta_s)$$

This leads to the following three equations, when the values nh, (n - 1)h and (n - 2)h are used for t:

$$D_n = K_1 e^{A_1 nh} + K_2 e^{A_2 nh} + f(\delta_{s_n})$$

$$D_{n-1} = K_1 e^{A_1 (n-1)h} + K_2 e^{A_2 (n-1)h} + f(\delta_{s_{n-1}})$$

$$D_{n-2} = K_1 e^{A_1 (n-2)h} + K_2 e^{A_2 (n-2)h} + f(\delta_{s_{n-2}})$$

This is a set of three linear equations in the three unknowns D_n, $K_1 e^{A_1 nh}$ and $K_2 e^{A_2 nh}$. It was solved for D_n in terms of D_{n-1}, D_{n-2}, $e^{A_1 h}$, $e^{A_2 h}$ and the function of δ_s evaluated at δ_{s_n}, $\delta_{s_{n-1}}$, and $\delta_{s_{n-2}}$.

For h small enough, the exponentials can be approximated. The result is the equation:

$$D_n - D_{n-1} = (D_{n-1} - D_{n-2})(1 - \tfrac{5}{8} A_7 Sh) - h^2 (A_9 D_{n-1} + A_{11} S^2 \delta_{s_{n-1}}) \tag{3-11}$$

This differs from equation 3-4. An equation analogous to equation 3-4 and which expresses the form of equation 3-11 is the equation:

$$v_n - v_{n-1} = hf_{11}(v_{n-1}, x_{n-1}, \text{m.p.}, \text{v.p.}) \tag{3-12}$$

where $v = \dot{x}$. The correspondence comes from letting $(D_n - D_{n-1})/h = v_n$, etc. Thus the acceleration algorithm referred to in section 3.2.3 is now a function of position D as well as velocity $\dot{D}$. This new dependence is present in all the control situations for submarine dive.

The equations for Case 2, the command control situation, are derived by holding D at some fixed value. The formulas are most accurate when this fixed value is close to the average that D will actually have. Thus in a change in dive rate to some ${}_{o}D$ (ordered D), it is best to use ${}_{o}D/2$.

When D is fixed, we can say that for any given stern plane angle δ_s there is a value for $\dot{D}$ (${}_{o}\dot{D}$) at which $\ddot{D} = 0$. This gives an equation relating δ_s and ${}_{o}\dot{D}$. Using this equation, the input to the command control model can be either ${}_{o}\dot{D}$ or δ_s.

With D fixed, the linear equation which approximates equation 3-10 is easily solved for $\dot{D}_n - \dot{D}_{n-1} = hf_2(\dot{D}_{n-1}, \text{m.p., v.p.})$. $\dot{D}_n$ approaches the fixed value ${}_{o}\dot{D}$. This ${}_{o}\dot{D}$ or the corresponding value of δ_s must be part of the input. ${}_{o}D$ must also be part of the input. ${}_{o}D$ (ordered D) is the value of D at which $\dot{D}$ is returned to zero. This is much more important for dive angle than for course angle, because of the disastrous effects on submarine and simulator if D grows too large.

c. Submarine Dive: Case 3

In the development of control situations for other maneuvers, the instructor control equation is of the form $v_n - v_{n-1} = hf_3(\text{m.p., v.p.})$. For the submarine dive maneuvers, however, the equation is of the form usually associated with Case 4. That is,

$$\dot{D}_n = \dot{D}_o \text{ for } nh < f_4 (\text{m.p., v.p.})$$

$$\dot{D}_n = {}_{o}\dot{D} \text{ for } nh > f_4 (\text{m.p., v.p.})$$

where $\dot{D}_o$ is dive angle rate when $t = 0$.

One of the reasons for this is that, in a dive maneuver, depth is more significant than dive angle. Since depth rate is proportional to sin D, D is actually a velocity rather than a displacement, and $\dot{D}$ an acceleration rather than a velocity. Any dive maneuver must consist of two applications of the instructor control formulation in order to achieve a given dive angle, followed by two more such applications when it is time to return the dive angle to zero.

d. Submarine Dive: Case 4

The program control model changes the dive angle abruptly from its original value D_o to its ordered value ${}_{o}D$. The appropriate time delay is found by integrating the depth change the submarine undergoes during two consecutive applications of the instructor control formulation. In the first of these, $\dot{D}$ goes from $\dot{D}_o$ to ${}_{o}\dot{D}$, where $\dot{D}_o$ is usually zero. In the second, it goes from ${}_{o}\dot{D}$ to zero in such a way as to make the final value of D equal to ${}_{o}D$.

Note that the input for both the instructor and program control models consists of D_o, $\dot{D}_o$, ${}_{o}D$, and ${}_{o}\dot{D}$.

3.3.2 Surface Vessel

The document named in Reference 2 contains the dynamic differential equations for the surface vessel. These equations contain constants a_1 through a_8. The equations hold for all normal, single-hulled surface vessels. Their applicability to hydrofoils and double-hulled vessels has not been established. Different surface vessels will have different values for the constants a_1 through a_8. Reference 2 contains values of these constants for a number of different surface vessels. These values are listed in Appendix C.

The surface vessel dynamic equations are

$$\dot{y} = (a_1\alpha + a_2\delta)V^2 + a_3Vy \tag{3-13}$$

$$\dot{\alpha} = (a_4\alpha + a_5\delta)V + a_6y \tag{3-14}$$

$$\dot{V} = a_7(a_4\alpha + a_5\delta)^2V^2 + a_8(V_o{}^2 - V^2) \tag{3-15}$$

The units are feet, radians, and seconds. y is rate of change of ship's heading angle, α is side-slip angle, course angle change ($\dot{C}$) is $y - \dot{\alpha}$, δ is rudder deflection angle, and V is speed. V_o corresponds to the $_oS$ of the submarine formulation. It is the speed for which the engine is set, or ordered speed. In the derivations, V_1 is used to denote the speed at the start of a maneuver.

3.3.2.1 Speed

The differential equation for speed can be solved directly only if the term $(a_4\alpha + a_5\delta)^2V^2$ is approximated in some simple way. According to equation 3-14, $(a_4\alpha + a_5\delta)V = \dot{\alpha} - A_6y$. In any steady turn, $\dot{\alpha}$ goes to zero and y goes to $_o\dot{C}$, the ordered course rate. The first assumption then, is that $(a_4\alpha + a_5\delta)V$ goes quickly to $-a_{6o}\dot{C}$. Thus, the accuracy of the simulation of speed in a turn depends on the relative size of the portion of the turn for which δ, and therefore $_o\dot{C}$, is fixed.

With this assumption, equation 3-15 can be solved directly and the solution put into the appropriate difference equation form by letting $e^{kh} = 1 + kh$ for the sufficiently small iteration interval, h.

The integral of V used to find the models for the instructor control and program control situations involves the approximation of $\log\left(\frac{V_f + V_o}{2V_f}\right)$. By making the assumption of positive values for all velocities, a series expansion of this function can be used.

If V is held constant in equations 3-13 and 3-14, then they become a pair of simultaneous linear first-order differential equations. As such they are easily solved. The solution technique constrains the rudder angle to change as a linear function of time. The expression used for this is $\delta = r_1 + r_2 t$.

The difference equations are found by letting $e^{kh} = 1 + kh$ for kh smaller than one. The current value of V is used in the difference equations.

The integral of $\dot{C}$ (see equations 3-1 and 3-2) is found as follows. First the formula for $\dot{C}$ is written with $\delta = r_2 t$ (i.e., $r_1 = 0$) and integrated from t = 0 to t = T, where $T = \underline{\delta}/r_2$. $\underline{\delta}$ is the principal value of δ in the turn (see section 3.3.1.2), and T is the time it takes for the rudder to move to that angle.

Then the formula for $\dot{C}$ is written with $\delta = \underline{\delta}$ (i.e., $r_1 = \underline{\delta}$ and $r_2 = 0$) and integrated from t = T to t = x. The equations for the models for the instructor and program control situations are found by using this integral.

The command control difference equations are very similar to the difference equations for the operator control situation, when $r_2 = 0$. A time delay is used for the time when δ is changing. The theoretical value for this time delay gave results that were consistently smaller than Case 1 by about 10 percent or more, so another was found by trial and error. The new time delay gave good results for the seven cases tested (see Appendix C). It was therefore incorporated into the command control model (see section 5.3.2.2).

3.3.3 Aircraft

The motion of an aircraft is described as that of a rigid body moving in space in a way that is somewhat restricted in its freedom of movement. As a point, it can accelerate or decelerate in the direction of motion, climb or dive at a limited angle, and turn in a circular arc. As a rigid body, its pitch angle will vary with its speed and climb rate, and its roll angle will vary with its turn rate. The order of causality is not fixed, but there is a separation of the six degree-of-freedom variables into two mutually independent but internally dependent categories. The development of the models for the two categories differs significantly. The categories will be called motion in the vertical plane and turn. Equations in the first category relate speed, climb rate, pitch angle, and thrust. In the second, they relate roll angle and turn rate as a function of speed.

3.3.3.1 Motion in the Vertical Plane

a. Sources

The equations in this category were developed to fit curves of speed versus climb rate, pitch versus climb rate, and thrust versus speed. Curves were available for each of these dependencies.

engine setting, altitude, and aircraft weight. In addition, there is a set of speed-versus-time curves varying according to the same parameters, which were used to check the acceleration and deceleration equations. These curves are all found in Reference 3. Of some help in fitting these curves were certain equations found in Reference 4. In addition, certain curves showing the interdependence of various aerodynamic coefficients at various Mach numbers were used. These are found in Reference 5.

b. Development

The acceleration equations are the only ones in this category that are developed to any degree of completeness. The curves of thrust versus airspeed are approximated by hyperbolas. These lead to simple differential equations for speed. Attempts have been made to express the parameters of the hyperbolas as functions of altitude.

Equations were set up to fit the curves of climb rate versus speed and climb rate versus pitch angle. It was necessary to postulate some algorithm for changing only one of these quantities since the interdependence equations cause the others to change accordingly. It was decided to change the pitch angle in such a way that the g-force experienced by the pilot would remain constant until the desired pitch angle was attained. Alternate algorithms might use a constant rate of change of pitch angle or a constant rate of change of climb rate. These possibilities were not explored because of the lack of information about the realism of any one of the algorithms as opposed to any of the others.

3.3.3.2 Turn

The equations for turn are completely geometric. They depend on the characteristics of the aircraft for only one parameter. The assumption is made that the turn is coordinated throughout its duration. This means that there is no lateral slippage, so whatever the airspeed, v, and the turn rate, ω, the following equation will hold

$$\frac{\omega v}{g} = \tan \phi \qquad (3\text{-}16)$$

where ϕ is the bank or roll angle of the aircraft and g is the acceleration due to gravity at that point.

The turn is accomplished as follows.

1. The order for course change and turn rate is given.
2. The aircraft rolls to the appropriate bank angle at the appropriate roll rate. The former depends on speed and ordered turn rate, the latter on the aircraft's characteristics.

change is the ordered course change minus the course change while the aircraft rolls out of the turn.

4. The aircraft rolls out of the turn.

3.4 TESTING AND EVALUATION

Once the formulas for the kinematic models were developed, they had to be tested. This was necessary to refine the models and to determine whether or not they could accomplish their purposes. Accuracy must increase as computer requirements increase. Computer requirements must decrease as accuracy decreases.

Appendix C contains at least one response curve for each of the kinematic models, with the exception of aircraft climb (see section 3.4.3). Wherever possible, the outputs from the simple dynamic equations is indicated on the graphs, as is any tactical trial data available for the same maneuvers. The emphasis, however, is on comparison with the dynamic equations. This is because no more accuracy can be expected from the kinematic equations than is inherent in the dynamic equations from which they are derived. Any closer fidelity to the actual tactical trial data on the part of the kinematic models is accidental. This would not be true if the kinematic models were modified in any way to conform to tactical trial data. Such an undertaking was beyond the scope of this study, however, so the most significant comparison remains that between the dynamic and kinematic models.

The kinematic equations used to generate the response curves appear in Appendix B. They will be referred to by section rather than by equation number, since an extensive set of equations were used for each set of response curves. References to specific equations for each curve are in section 4.2.

The curves will be discussed in Sections IV and V.

3.4.1 Kinematic Response Curves: Submarine

All submarine maneuvers were run for a submarine of the SS(B)N598 class. Reliable coefficient data was not available for the other submarines (see section 3.5).

3.4.1.1 Speed

Figure C-1 is an acceleration from 2 knots to 15 knots and Figure C-2 a deceleration from 20 knots to 2 knots. Both show response curves for all four cases as well as points generated by the dynamic equations (Reference 1). The formulas appear in section B.1.2.1 of Appendix B.

... entering the turn is 10 knots. The formulas used are those in section B.1.2.2 of Appendix B. The dynamic data points that appear on the graph are from Reference 6.

3.4.1.2 Turn

a. Response Curves

Figure C-3 is for the same turn maneuver as Figure C-4: 10 knots and 20 degrees rudder. Figure C-3 shows course angle, course angle rate, and position. All the equations used are from section B.1.2.2 of Appendix B. These equations are for speed as well as rate of change of course. Speed in a turn can be generated without any reference to the course rate equations. Equations for course rate, however (Cases 1 and 2), use current speed at every iteration. Thus all the equations in Appendix B, section B.1.2.2 are used, plus those in section B.3.1 for position updating.

Dynamic data points come from Reference 6.

b. Advance, Transfer, and Tactical Diameter

Figures C-5, C-6, and C-7 provide a good way of evaluating the accuracy of the kinematic equations for submarine turn. Reference 1 contains a comparison between certain sets of data generated by the simple dynamic equations and the corresponding tactical trial data. This data consists of turning characteristics for nine different turns. Speeds entering the turns are 5, 10, and 20 knots, and rudder angles are 10°, 20°, and 30°. The various combinations of these quantities provide data on a total of nine turns.

Advance, transfer, tactical diameter, time to turn 360 degrees, speed in turn, and average course change per minute are the quantities compared. Advance is the component of the displacement of the vessel, when it has changed its course by 90 degrees, which is in the direction of its motion before entering the turn. Transfer is the component of this displacement perpendicular to the original direction of motion. Tactical diameter is the perpendicular distance between the original and final directions of motion when it has changes its course by 180 degrees.

These same quantities are generated for the same nine turns by the kinematic equations. Case 4 is used for two reasons. First, it is the least accurate; this gives a worst-case evaluation. If the kinematic equations give satisfactory results for Case 4, they should give even more satisfactory results for the other cases.

Second, if Case 4 is used, all the necessary quantities can be calculated by hand in a few minutes. Use of any of the other cases would require an extensive computer program. Case 4 provides two time delays. One is between the time at which the maneuver is initiated and the time at which the speed is changed from ${}_oS$ to S_f. The other is between the initiation time and the time at which the course rate is changed from zero to ${}_o\dot{C}$. The necessary formulas are derived from these time delays.

Let T_s be the time delay for speed and T_c the time delay for course rate.

Consider first the case where $T_c < T_s$. The turn maneuver will be performed in three stages. While $t < T_c$, the submarine is going straight ahead. While $T_c \leq t < T_s$, it is turning with speed oS and turn rate $o\dot{C}$. During this interval it turns through an angle $(T_s - T_c)\, o\dot{C}$ with turning radius $\frac{oS}{|o\dot{C}|}$. When $T_s \leq t$, it is turning with speed S_f and turn rate $o\dot{C}$. The radius of this $\frac{S_f}{|o\dot{C}|}$.

When the submarine has turned $\pi/2$ radians:

$$\text{Advance} = oST_c + \frac{oS}{/o\dot{C}/} \sin\left[(T_s - T_c)/o\dot{C}/\right] + \frac{S_f}{/o\dot{C}/}\left(1 - \cos\left[\frac{\pi}{2} - (T_s - T_c)/o\dot{C}/\right]\right) \quad (3\text{-}17)$$

$$\text{Transfer} = \frac{oS}{/o\dot{C}/}\left[1 - \cos\left[(T_s - T_c)/o\dot{C}/\right]\right] + \frac{S_f}{/o\dot{C}/} \sin\left[\frac{\pi}{2} - (T_s - T_c)/o\dot{C}/\right]$$

providing $(T_s - T_c)\,/o\dot{C}/ < \frac{\pi}{2}$ (3-18)

If $(T_s - T_c)\,/o\dot{C}/ > \frac{\pi}{2}$

$$\text{Advance} = ST_c + \frac{oS}{/o\dot{C}/}$$

$$\text{Transfer} = \frac{oS}{/o\dot{C}/}$$

These equations can also be derived from 3-17 and 3-18 by letting $(T_s - T_c)/o\dot{C}/ = \pi$.

Thus for $T_c \leq t < T_s$

$$\text{Advance} = oST_c + \frac{oS}{/o\dot{C}/} \sin\left[(T_s - T_c)\, o\dot{C}\right] + \frac{S_f}{/o\dot{C}/}\left(1 - \cos\left[\frac{\pi}{2} - (T_s - T_c)/o\dot{C}/\right]\right)$$

$$\text{Transfer} = \frac{oS}{/o\dot{C}/}\left[1 - \cos\left[(T_s - T_c)/o\dot{C}/\right]\right] + \frac{S_f}{/o\dot{C}/} \sin\left[\frac{\pi}{2} - (T_s - T_c)\,|o\dot{C}|\right]$$

with $(T_s - T_c)/o\dot{C}/$ equal to $\min\left(\frac{\pi}{2}, (T_s - T_c)/o\dot{C}/\right)$

$$\text{Tactical Diameter} = \text{Transfer} + \frac{S_f}{/o\dot{C}/} \quad (3\text{-}19)$$

$$\text{Time to Turn } 360^\circ = \frac{360^\circ}{/o\dot{C}/} + T_c \quad (3\text{-}20)$$

$$\text{Average Turn Rate} = 360^\circ/\text{Time to Turn } 360^\circ \quad (3\text{-}21)$$

Advanced and transfer equations can be simplified by letting $\sin(T_s - T_c)\,{}_o\dot{C} = (T_s - T_c)\,{}_o\dot{C}$. The angles are small enough so that any error will be very small compared to the dominant terms in the equation. Tables C-1, C-2, and C-3 of Appendix C list the

following six quantities for the nine turns under discussion.

Advance
Transfer
Tactical Diameter
Time to Turn 360^{o}
Degrees per Minute
Final Speed for the Nine Turns

Table C-1 relates to the actual submarine, Table C-2 to the dynamic equations, and Table C-3 to Case 4 of the kinematic equations.

Consider now the case where $T_s < T_c$. While $o \leq T_c$ the submarine moves in a straight line; for $T_c \leq t < T_s$ it turns with radius $\frac{S_f}{/o\dot{C}/}$

Thus

$$\text{Advance} = oST_s + S_f (T_c - T_s) + \frac{S_f}{/o\dot{C}/} \qquad (3\text{-}2.$$

$$\text{Transfer} = \frac{S_f}{/o\dot{C}/} \qquad (3\text{-}2.$$

$$\text{Tactical Diameter} = \frac{2S_f}{/o\dot{C}/}$$

Where $/o\dot{C}/$ is in radians. Equations 3-19, 3-20 and 3-21 are still valid.

Figures C-5, C-6, and C-7 of Appendix C display advance, transfer, and tactica diameter as generated by these equations. The same quantities as generated by the dynamic equations and the actual submarine are also shown on the graphs.

3.4.1.3 Dive

Response curves are presented in Appendix C for two dive maneuvers. In the first, shown in Figure C-8, the stern plane angle is changed from zero to -15° in three seconds, held at -15° for another eighteen seconds, and then changed to +25° at the rate of 5° per second. The speed of the submarine is 20 knots. This is run for all four cases of the submarine kinematic equations. In the equations for Case 3, $_0D$ was 10° (i.e., the real ordered dive angle divided by two; This is explained in section A.1.4.2.2 of Appendix A.) For Case 4, $_0D$, the real ordered dive angle, is 20°. This is because the maneuver is an overshoot maneuver, and the stern plane was made to begin moving when D reached 20°. $_0\dot{D}$ for Case 4 is obtained from the figures for Case 3 (i.e., using $\delta_s = -15°$ and $_0D = 10°$). The formulas used for the kinematic models are in section B.1.2.3 of Appendix B. Both the tactical points and the points from the dynamic response curves came from Reference 1.

The response curves for the second dive maneuver are displayed in Figure C-9 of Appendix C. This is a dive and level-up maneuver. The stern plane angle changes too often for the terms $_0D$ and $_0\dot{D}$ to have any meaning; accordingly, Cases 3 and 4 are not used. The formulas for Case 2 are used with $_0D$ equal to zero. Once again, the kinematic equations are in section B.1.2.3 of Appendix B, and the tactical and dynamic data points are from Reference 1.

3.4.2 <u>Kinematic Response Curves: Surface Vessel</u>

The surface vessel runs are all of the same type. The rudder is moved to the desired rudder angle as a linear function of time, then held at that angle for the duration of the run. This is done for several values of speed-entering-the-turn and for several rudder angles for each of three surface vessels. The surface vessels are a destroyer of the DD-445 class, a cruiser of the CA-68 class, and a long-hull destroyer of the DD-692 class.

The only available response curve generated by the dynamic equations is for a 14.7-degree rudder angle turn at 24 knots for the DD-445 (Reference 7). The curves for course change, course change rate, and trajectory for this maneuver are shown in Figure C-10 of Appendix C. Figure C-10 also contains points representing tactical trial data for this maneuver (again from Reference 7). The speed change in the turn for this maneuver (kinematic and actual) is shown in Figure C-13.

Figures C-11 and C-12 show course change, course change rate, and trajectory for two more DD-445 turns. The first enters the turn at 15 knots and uses a 10-degree rudder; the second enters the turn at 34.4 knots and uses a 33.3-degree rudder. Tactical data points come from Reference 8. The speed in the turn for these two maneuvers is shown in Figure C-13.

Figures C-14 and C-15 show course change, course change rate, and trajectory for two DD-692 turns. They are at 33 knots and 10-degree rudder and at 15 knots and 25-degree rudder, respectively. Speeds during both turns are shown in Figure C-16. Tactical data points all come from Reference 10.

The course and trajectory curves for the two CA-68 turns are in Figure C-17 and C-18. The turn in Figure C-17 is entered at 15 knots and uses a 14.5-degree rudder angle. The one in Figure C-18 is entered at 32 knots and uses a 15-degree rudder. Reference 9 is the source of tactical data points. Speed changes are shown in Figure C-19. All surface vessel maneuvers are run using all four sets of kinematic equations. These equations appear in section B.2.2.2 of Appendix B. Position updating is accomplished using the formulas in section B.3.1 of Appendix B. A variety of maneuvers is used to show the variety of accuracy differences between the four cases and between the kinematic model and true tactical data.

3.4.3 Kinematic Response Curves: Aircraft

Only three curves were generated for the aircraft simulation models, and only one of these can be compared to any other response curves. Figure C-20 of Appendix C displays two aircraft acceleration maneuvers; both are at sea level with the same weight load on the aircraft. One is an acceleration from 260 knots to 580 knots, the other from 260 knots to 650 knots. Both use a five-second iteration interval, and are compared to Approved Performance Data from Reference 3.

Figure C-21 depicts bank angle and total course change versus time. Figure C-22 shows displacement perpendicular to the original direction of motion versus displacement parallel to the original direction of motion. Both are for a five-degree-per-second turn at 500 knots airspeed. On the curve, these displacements are called respectively transfer and advance. This is for the sake of brevity only, as they are not actually transfer and advance. The definitions of transfer and advance are in section 3.4.1.2. There were no performance curves available to compare with these curves.

Similarly, there were no performance curves available for any maneuvers in the vertical plane. Any curves that could be generated by the kinematic model for this aspect of aircraft motion would show nothing that could be used for evaluation purposes.

3.5 MODIFICATION OF SUBMARINE CONSTANTS

Reference 1 contains values for the constants of the submarine equations of motion (A_1 through A_{11}, section 3.3.1) for three submarines. Close inspection of these constants showed that there are mistakes among them. If the constants are used in the differential equations, the results are quite different from the results claimed in the reference. In particular, the constant relating speed entering a turn to speed during the turn was wrong for all three submarines; this was easy to correct. There was no obvious way, however, to correct the constants of the dive equation for the submarines Attack Center II and Attack Center III. The mistakes in these constants cause the dive angle to diverge very quickly; they serve to cast doubt upon the validity of all the constants for these two submarines. This doubt was reinforced by the zero value for A_5 for one of them and the same value for A_4 for the other. For this reason, the turn equations were not run for either.

3.5.1 Speed During Turn

The constant A_2, relating ${}_0S$ to S_f by the formula $S_f = {}_0S/(1 + A_2|\underline{\delta_r}|)$, was corrected by taking values of ${}_0S$, S_f and $\underline{\delta_r}$ for several turns. This was done for all three submarines. For the first two, only the position of the decimal point had to be changed. The value of A_2 for Attack Center III however, had to be changed from 0.0403 to 0.0271.

3.5.2 Dive

Section A.1.4 of Appendix A contains a discussion of the differential equation for dive. In this discussion, it is pointed out that certain values of the constants will lead to unstable behavior of the submarine during an attempted dive. The curve of $\ddot{D}$ versus $\dot{D}$ has a maximum and a minimum. It is shown in section A.1.4 that if the $\dot{D}$ axis is not between the two, a small deflection of the stern plane angle will lead to a divergent pitch rate.

The maximum and minimum are at $\dot{D} = +A_7S/2A_8$ and $\dot{D} = -A_7S/2A_8$, respectively. Substituted into equation 3-10, this gives the maximum and minimum points as, respectively (since A_7 is positive and A_8 negative),

$$\left(\frac{A_7S}{2A_8}\right),\ -\left(\frac{(A_7S)^2}{4A_8}+A_9D+A_{11}S^2\delta_s\right)\ \text{and}\ \left(\frac{-A_7S}{2A_8}\right),\ -\left(\frac{-(A_7S)^2}{4A_8}+A_9D+A_{11}S^2\delta_s\right).$$

If the $\dot{D}$ axis is to separate these two points, then the inequality

$$\frac{(A_7S)^2}{4A_8} < A_9D + A_{11}S^2\delta_s < -\frac{(A_7S)^2}{4A_8}$$

must hold. Expressed otherwise,

$$|A_9D + A_{11}S^2\delta_s| < \left|\frac{(A_7S)^2}{4A_8}\right| \tag{3-24}$$

Inequality 3-24 can now be used as a rough estimate of the validity of the constants. The values of A_9 and A_{11} were almost identical for the three submarines, so they will be assumed valid and values A_7 and A_8 examined. A rough check is to let D = 0. This changes the inequality to

$$A_{11}|\delta_s| < -\frac{A_7^{\,2}}{4A_8}$$

For Attack Center I , $|\delta_s| < 15°$

For Attack Center II , $|\delta_s| < 0.000003°$

and Attack Center III, $|\delta_s| < 0.000625°$

As the actual values of $\dot{D}$ and $\ddot{D}$ approach the maximum (or minimum point), D grows with a sign opposite to that of δ_s. Thus it is possible for $|\delta_s|$ to be greater than 15° for Attack Center I while $|A_9D + A_{11}S^2\delta_s|$ is still less than $-(A_7S)^2/4A_8$. Where $-A_7^{\,2}/4A_8$ is very small, however, this effect is of very little help. Consider the constraints on D for $\delta_s = -10°$ and S = 10 knots.

Attack Center I , $-20.6° < D < 103°$

Attack Center II , $47.54534° < D < 47.54536°$

Attack Center III, $95.0848° < D < 95.0966°$

The impossibility of attaining the conditions for Attack Centers II and III is obvious. If they are not attained when $\dot{D}$ and $\ddot{D}$ are near the maximum or minimum points on the graph, then D will grow without limit (see section A.1.4 in Appendix A).

Thus, A_7 and/or A_8 are completely erroneous for Attack Centers II and III.

3.6 COMPUTER REQUIREMENTS

The kinematic models presented in this report are of a general nature, and may be programmed on a great variety of digital computing devices. This being so, an analysis of running time or computation time per iteration for running the kinematic equations on any one particular computer is of less interest than an analysis of the number of operations of various kinds necessary to perform the mathematics involved. All computer operations are therefore put into four categories: transfer operations, addition, multiplication, and division. By transfer operations we mean to include data transfers as well as program transfers. Thus Clear and Add, Move, Transfer on Zero, and Store are all called transfer operations. Since they take the same time in most computers, this will not lead to any ambiguity.

Storage requirements are also given. This includes both fixed storage for the instructions, any necessary constants, and temporary storage. The storage requirements for each of the four cases are stated separately.

SECTION IV

RESULTS

4.1 INPUT AND OUTPUT

Appendix B of this report contains a list of equations and the rules for their use. These equations are designed to describe the motion of three types of vehicles. There are certain inputs necessary for this description, consisting of vehicle parameters (v.p), maneuver parameters (m.p.) and initialization information.

Another type of input is also necessary when a submarine or surface vessel is to be simulated. For every maneuver of one of these vehicles, there are four possible simulation models. Before any simulation can be performed, a choice must be made as to which of the four models will be used.

In order to make this choice, the nature of the four models and their differences must be known and understood. Consider the input and output of each.

4.1.1 Inputs to the Four Kinematic Models

Section 3.1.2 describes a tentative correspondence between the four models and four types of uses of simulation models. This correspondence is based on the form of the input from the four types of uses and on the expected accuracy of the outputs from the four kinematic models. The models have much more flexibility with regard to input than is implied in that section, however.

4.1.1.1 Acceleration/Deceleration

The equations for this maneuver are shown in sections B.1.2.1 and B.2.2.1 of Appendix B. It is obvious from these equations that the inputs to all four models (or rather all three, since Cases 1 and 2 are identical) are the same.

4.1.1.2 Turn

These equations, for submarine and surface vessel, are listed in sections B.1.2.2 and B.2.2.2 of Appendix B. The input to Case 1 requires the current value of the rudder angle; the input to the other three cases do not. The input to Case 1 also requires $\underline{\delta}$ or $\underline{\delta_r}$ (see section 3.3.1.2), whereas the input to the other three cases require ${}_o\dot{C}$. All four cases require V_o or ${}_oS$, however, and with ${}_oS$ or V_o known, ${}_o\dot{C}$ can be found from $\underline{\delta_r}$ or $\underline{\delta}$ and vice versa.

$$S_f = f({}_oS, {}_o\dot{C}) \qquad \text{(Equation B-7)}$$

$$\underline{\delta_r} = f({}_oS, S_f) \qquad \text{(Equation B-9)}$$

Thus, $\underline{\delta_r} = f({}_oS, {}_o\dot{C})$

Furthermore, knowing T, $\delta_r = t\underline{\delta_r}/T$

Conversely, $${}_o\dot{C} = f({}_oS, \underline{\delta_r}) \qquad \text{(Equation B-8)}$$

While for the surface vessel,

$$V_f = f(V_o, {}_o\dot{C}) \qquad \text{(Equation B-44)}$$

$$\underline{\delta} = f(V_f, {}_o\dot{C}) \qquad \text{(Equation B-40)}$$

Thus, $\underline{\delta} = f(V_o, {}_o\dot{C})$

Furthermore, knowing T, $\delta = t\underline{\delta}/T$

Conversely, $$V_f = f(V_o, \underline{\delta}) \qquad \text{(Equation B-43)}$$

$${}_o\dot{C} = f(V_f, \underline{\delta}) \qquad \text{(Equation B-40)}$$

Thus, $${}_o\dot{C} = f(V_o, \underline{\delta})$$

In other words, if considerations of accuracy, computer requirements, or the form of the output dictate, any one of the four models can use any form of input that can be used by any other of them. Exceptions to this occur when the response to repeated turn maneuvers in close succession or combination turn and acceleration/deceleration maneuvers are being simulated. In those situations, Case 1 is used. Cases 2, 3 and 4 could probably be adapted to simulate these maneuvers, but such an adaption is not investigated in this report.

4.1.1.3 Dive

In the case of the dive equations (section B.1.2.3, Appendix B), there is a genuine difference between the kinds of input that can be accepted by them. The input to Case 1 is δ_{s_n}, the current value of the stern plane deflection angle along with D_o and D_1. The input to case 4 is D_o and $\dot{D}_o$, the initial dive angle and dive angle rate, and ${}_oD$ and ${}_o\dot{D}$ the ordered dive angle and dive angle rate.

Cases 2 and 3 are transitional steps between 1 and 4. Case 2 can accept δ_n as input, using Equation B-25 from Appendix B. If Equation B-24 is used for Case 2, then the input is $_o\dot{D}$ and the use of the formula has to be monitored, using $_oD$ or time. The same input is used for Case 3: $_o\dot{D}$ and $_oD$ or time. Case 4 cannot use time however; only $_o\dot{D}$ and $_o\dot{D}$. $\dot{D}_o$ and D_o can be used to initialize all four cases, where D_o and $D_o + h\dot{D}_o$ are used to initialize Case 1. Similarly, D_o and D_1 can also be used for all four cases, letting $\dot{D}_o = (D_1 - D_o)/h$.

4.1.2 Outputs from the Four Kinematic Models

The form of the outputs from the four models is shown in section 3.2.2, Figures 3-1 through 3-4. These figures show velocity versus time; velocity can be S, V, or $\dot{C}$. The output of the four cases of the dive simulator is somewhat different.

For the dive simulator, the dotted line representing $_o\dot{C}$ in Figures 3-1 through 3-4 is now a connected series of broken line segments representing roughly $_o\dot{D}$, which is a function of δ_s. For Case 1, $\dot{D}$ approaches this curve asymptotically after a delay in starting (as in Figure 3-1). The line segments move up or down, however, since the correspondence between δ_s and $_o\dot{D}$ depends on the current value of D.

For Case 2, the series of line segments is fixed and $\dot{D}$ starts toward it abruptly (as for $V \approx o$ in Figure 3-2), and changes direction abruptly as the $_o\dot{D}$ curve changes direction. The output from Case 3 resembles a combination of Figures 3-3 and 3-4, with D the ordinate rather than $\dot{D}$. That is to say, D increases linearly after an elapsed time delay.

The output from Case 4 resembles Figure 3-4, with D again the ordinate. In general the initial time delay is larger than it is for S, V or $\dot{C}$.

4.2 ACCURACY

In section 4.1 the differences in input and output were described for the four kinematic models. It can be seen from this description that, except for the submarine dive, we can always obviate the differences in the input requirements of the four cases. The differences in the form of the output are most obvious in the velocity response curves but much more subtle as these curves are integrated to give position.

It is necessary, therefore, that more permanent criteria be established for selection among the four cases. In Section II the question of the existence of the new models is asked in terms of their fidelity and computer requirements. The computer requirements of the new models are described in section 4.3. The paragraphs below describe their accuracy.

and shown in Appendix C. Evaluation is both qualitative, from a visual examination of the graphs, and quantitative, using certain percent errors. Case 1, with a one-second iteration interval, is used as the standard for quantitative comparison whenever dynamic data is not available. All equation references are for Appendix B.

4.2.1 Acceleration/Deceleration

Quantitative evaluation is on the basis of percent of time loss or gain in the maneuver. The difference in distance covered between the run being examined and the standard run is divided by the final speed to give the time difference. This is then divided by the approximate time it took for the acceleration/deceleration to be completed, using the standard model. Speed loss in a turn is treated as a deceleration.

4.2.1.1 Submarine (Tables 4-1, 4-2 and 4-3)

Graphs for submarine acceleration/deceleration maneuvers are in Figures C-1, C-2 and C-4 of Appendix C. For all three, Cases 1 and 2 follow the points generated by the dynamic equations very closely. The total distance covered using Cases 3 and 4 should be identical because of the way in which they are derived. The major difference is between Cases 1 and 2 and Cases 3 and 4. By inspection, this difference seems quite small for the acceleration (Figure C-1) and speed loss in the turn (Figure C-4), but quite large for the deceleration (Figure C-2).

An iteration interval of one second was used for all the cases listed. The points generated using a ten-second iteration interval are seen by inspection to be very close to those generated using a one-second interval.

TABLE 4-1. SUBMARINE ACCELERATION

Acceleration, 2 knots to 15 knots; Time, 150 seconds					
Case	Equations	Distance	Δ Distance	ΔTime	% Δ Time
1, 2	1	1037 yards	--	--	--
3, 4	2	1005 yards	-32 yards	-3.8 seconds	3

TABLE 4-2. SUBMARINE DECELERATION

Deceleration, 20 knots to 2 knots; Time, 1200 seconds					
Case	Equations	Distance	Δ Distance	Δ Time	% Δ Time
1, 2	1	2621 yards	--	--	--
3, 4	2	2155 yards	-466 yards	-414 seconds	35

TABLE 4-3. SPEED LOSS IN SUBMARINE TURN

Speed Loss in Turn, 10 knots to 6.43 knots; Time, 110 seconds					
Case	Equations	Distance	Δ Distance	Δ Time	% Δ Time
1, 2	9, 12	444 yards	--	--	--
3, 4	7, 14	450 yards	+6 yards	+1.7 seconds	2

4.2.1.2 Surface Vessel (Table 4-4)

Graphs of speed loss in a surface-vessel turn are shown in Appendix C for the surface vessels DD-445 (Figure C-13), DD-692 (Figure C-16) and CA-68 (Figure C-19).

The only one for which points were available from the dynamic equations was the turn at 14.7° rudder for the DD-445. These points are not shown on the graph, but they coincide exactly with the points generated by the kinematic equations.

For most of the turns, there is no discernible difference between the velocities generated by Cases 1 through 4. Differences show up only when $V_o - V_f$ is large enough. When a difference appears, it always displays the same behavior; Cases 3 and 4 give too rapid a speed loss.

A quantitative breakdown is given for the maneuver where this difference is most extreme, the turn at 15° rudder angle and 32 knots' speed entering the turn of the surface vessel CA-68. Inspection of the other graphs show the difference between the cases to be roughly proportional to $V_o - V_f$.

The value for h, the iteration interval, is given in seconds; it is one second, unless specified otherwise. All distances are in feet, and time in seconds.

TABLE 4-4. SPEED LOSS IN SURFACE VESSEL TURN

32 knots to 26 knots; Time, 180 seconds					
Case	Equations	Distance	Δ Distance	Δ Time	% Δ Time
1	45, 46, 47	8751	--	--	--
1 (h=5)	45, 46, 47	8700	-51	-1.[illegible]	0.6
2	49, 50, 51	8715	-36	-0.8	0.5
3, 4	44, 53	8395	+355	+8.1	5.0

4.2.1.3 Aircraft

Figure C-20 in Appendix C shows the points generated by the kinematic equations for aircraft acceleration, using an iteration interval of five seconds; this is for acceleration at sea level. The results, especially for the lower ordered speed, are quite good. The error in the case of the higher ordered speed demonstrates one of the limitations of the aircraft kinematic speed equations. For higher values of thrust, the assumption that thrust remains constant for a given throttle setting as speed changes becomes invalid. For low altitudes, however, this is still a fairly good assumption.

Not shown are the results of the acceleration simulation at 35,000 feet. In that case, the curve of drag versus airspeed resembles a hyperbola only for airspeeds less than 570 knots. Therefore only accelerations to low ordered speeds were simulated. The results of these simulations showed about 50 percent error.

4.2.2 Turn

The quantitative evaluation of the models for simulating a turn is based mainly on the shape and size of the turning circle. The quantities advance, transfer, tactical diameter, and time to turn 180° are compared. These quantities are compared for Cases 1 through 4 and also for two models very much like the original unmodified kinematic models. These two have the vehicle change directly from straight line motion to a circular turn at the instant that the turn order is given. In the first, the turn radius is $S_f / |{}_0\dot{C}|$; in the second, it is ${}_0S / |{}_0\dot{C}|$.

Wherever possible, tactical trial data values for these quantities are also compared.

4.2.2.1 Submarine (Tables 4-5 and 4-6)

Submarine runs were for the SS(B)N 598 only. A full turn simulation was run for only one turn, at 20° rudder and 10 knots entering the turn. This is shown in Figure C-3 of Appendix C. The following table lists advance, transfer, tactical diameter, and time to turn 180° for various ways of simulating this maneuver. Equations 6, 8, 9, 57, 58, 59 and 60 of Appendix B were used for all the cases of the kinematic models. The other equations used for each case are listed in the column for equations. Iteration interval is one second, except where stated otherwise. All distances are in yards, all time in seconds.

Figures C-5, C-6 and C-7 show respectively advance, transfer, and tactical diameter for nine turns, using Case 4 only. Equations 16 and 18 of Appendix B were used to find the appropriate time delays; Equations 3-22, 3-23 and 3-17 of section 3.4.1.2 were used to find advance, transfer and tactical diameter. These equations also required $\dot{C}_o$ and S_f. These were found using Equations 8 and 9 from Appendix B because the nine turns were described in terms of rudder angle and speed entering the turn.

Appendix C contains tables of advance, etcetera, for Case 4 of the kinematic equations (Table C-III), the dynamic equations (Table C-II), and the actual tactical trial data (Table C-I). Percent errors are listed in Table 4-6, using the dynamic data as a standard. Time in this case is time to turn 360°.

All distances are in yards, all time in seconds.

4.2.2.2 Surface Vessel (Tables 4-7, 4-8, and 4-9)

Section 3.4.2 describes seven surface vessel turns whose response curves appear in Appendix C. Data generated by the dynamic equations is available for only one of these turns, the DD-445 turn with 14.7° rudder angle and a speed of 24 knots entering the turn. The response curves are shown in Figure C-10. Cases 1, 3 and 4 duplicate the course change almost exactly.

All seven graphs contain tactical trial data as well as the points generated by the four kinematic equations. By inspection of Figures C-10, C-11, C-12, C-14, C-15, C-17 and C-18, it can be seen that the actual turning circles are different from the kinematic turning circles. The size and direction of the difference is random, and of the same order of magnitude as that between the dynamic and kinematic turning circles. This can be seen from Figure C-10, where the difference between dynamic and actual turn rate is as great as any of those between kinematic and actual turn rates.

For several of the turns, the spread between the trajectories generated by the various kinematic models is very small. Taking Case 1 with a one-second iteration interval as the standard, advance has percent spread shown in Table 4-7 (this includes all cases and iteration intervals).

TABLE 4-5. SUBMARINE TURN

Turn at 20° Rudder Angle, Speed 10 Knots Entering Turn													
Case Equations	Advance	Δ Adv.	% Δ	Transfer	Δ Trans.	% Δ	Tactical Diameter	Δ T. D.	% Δ	Time to Turn 180°	Δ Time	% Δ
Dynamic Data	285	--	--	196	--	--	380	--	--	175	--	--
1 10, 11	310	+ 25	9	205	+ 9	5	391	+ 11	3	182	+ 7	4
2 12, 13	310	+ 25	9	205	+ 9	5	391	+ 11	3	182	+ 7	4
2(h=10) 12, 13	269	- 16	6	193	- 3	2	382	+ 2	0.5	173	- 2	1
3 14, 15	400	+115	40	207	+11	6	388	+ 8	2	208	+33	19
4 16, 17, 18	409	+124	44	181	-15	8	362	- 18	5	208	+33	19
$R = S_f / \lvert {}_o\dot{C} \rvert$	181	-104	36	181	-15	8	362	- 18	5	158	-17	10
$R = {}_oS / \lvert {}_o\dot{C} \rvert$	281	- 4	1	281	+85	43	562	+182	48	158	-17	10
Tactical Trial Data	270	- 15	5	220	+24	12	400	+ 20	5	181	+ 6	3

TABLE 4-6. NINE SUBMARINE TURNS, CASE 4

	Entering Speed	Rudder Angle	Time for 360°		Advance		Transfer		Tactical Diameter	
			Δ	%Δ	Δ	%Δ	Δ	%Δ	Δ	%Δ
Kin	5 knots	10°	+44	5	+ 84	20	-24	8	-27	4
T. T.	5 knots	10°	-79	9	+ 37	9	-53	17	-51	8
Kin	5 knots	20°	+66	10	+120	42	-17	9	-18	5
T. T.	5 knots	20°	-26	4	+ 2	0.7	+ 2	1	0	0
Kin	5 knots	30°	+99	17	+153	69	-11	7	-12	4
T. T.	5 knots	30°	-17	3	-122	55	-11	7	-30	10
Kin	10 knots	10°	+17	4	+ 87	20	-24	8	-27	4
T. T.	10 knots	10°	+19	4	- 33	8	+42	13	+39	6
Kin	10 knots	20°	+34	10	+124	67	-17	9	-18	5
T. T.	10 knots	20°	+ 5	2	- 15	5	+21	11	+20	5
Kin	10 knots	30°	+51	17	+157	69	-12	8	-12	4
T. T.	10 knots	30°	- 8	3	+ 9	4	+13	9	-10	3
Kin	20 knots	10°	+ 9	4	+ 93	22	-20	6	-27	4
T. T.	20 knots	10°	- 4	2	- 39	9	- 4	1	+ 9	2
Kin	20 knots	20°	+20	12	+132	46	-12	6	-13	4
T. T.	20 knots	20°	+ 5	3	+ 57	20	+27	14	+45	12
Kin	20 knots	30°	+32	22	+163	69	- 2	1	+ 1	0.4
T. T	20 knots	30°	-56	39	+ 63	27	+18	13	+33	12

TABLE 4-7. SURFACE VESSEL TURNS,

DIFFERENCES IN ADVANCE

Figure	Vessel	Rudder Angle	Speed Entering	Percent Spread
C-10	DD-445	14.7°	24.0 knots	*
C-11	DD-445	10.0°	15.5 knots	3
C-12	DD-445	33.0°	34.4 knots	16
C-14	DD-692	10.0°	33.0 knots	4
C-15	DD-692	25.0°	15.0 knots	3
C-17	CA-68	14.5°	15.0 knots	26
C-18	CA-68	15.0°	32.0 knots	19

*No data for $C > 65°$.

Inspection of Figure C-10 shows that if the maneuver were continued to $C = 90°$ the percent spread in advance would be less than one percent. Figures C-10, C-11, C-14 and C-15, besides having a small spread, show only two discernible trajectory curves. All generate an equally accurate simulation of these models. An evaluation of the differences in the models is only meaningful for the maneuvers depicted in Figures C-12, C-17 and C-18. Figure C-17 and C-18 show the same juxtaposition of the various cases, so Figure C-17 will be used as a worst case.

In Tables 4-8 and 4-9, the same quantities will be compared for the maneuvers depicted in Figures C-12 and C-17 as were compared for the turn in Figure C-3 (see section 4.2.2.1). All iteration intervals are one second unless otherwise specified. Equations listed are from Appendix B. Equations 39, 40, 41, 42, 57, 58, 59, and 60 were used for all four cases. They are not listed separately for each.

All distances are in yards, all times in seconds. The columns listing time refer to the time consumed in turning 180°.

Case 1 uses Equations 45, 46, 47 and 48.

Case 2 uses Equations 48, 49, 50 and 51.

Case 3 uses Equations 43, 52 and 53.

Case 4 uses Equations 43, 54 and 55.

TABLE 4-8. SURFACE VESSEL TURN, DD-445

34.4 Knots, 33° Rudder Angle						
Case	**Advance**	**Δ Adv.**	**% Δ**	**Transfer**	**Δ Trans**	**% Δ**
1	708	--	--	418	--	--
2	820	+112	16	402	-16	4
3	708	0	0	433	+15	4
4	718	+ 10	1	390	-28	7
$R = S_f / \lvert {}_o\dot{C} \rvert$	392	-316	45	392	-26	6
$R = {}_oS / \lvert {}_o\dot{C} \rvert$	409	-299	42	409	- 9	2
Case	**Tactical Diameter**	**Δ T.D.**	**% Δ**	**Time**	**Δ Time**	**% Δ**
1	805	--	--	82	--	--
2	797	- 8	1	87	+ 5	6
3	821	+16	2	83	+ 1	1
4	781	-24	3	83	+ 1	1
$R = S_f / \lvert {}_o\dot{C} \rvert$	784	-21	3	66	-16	20
$R = {}_oS / \lvert {}_o\dot{C} \rvert$	818	+13	2	66	-16	20

TABLE 4-8. SURFACE VESSEL TURN, CA-68

15 Knots, 14.5° Rudder						
Case	Advance	Δ Adv.	% Δ	Transfer	Δ Trans.	% Δ
1	1168	--	--	735	--	--
1 (h=10)	1173	+ 5	0.4	720	-15	2
2	1075	- 93	8	727	- 8	1
2 (h=10)	1000	-168	14	717	-18	2
3	1310	+142	12	734	- 1	0.1
4	1300	+132	11	621	-114	16
$R = S_f / \lvert {}_o\dot{C} \rvert$	617	-551	47	617	-118	16
$R = S / \lvert {}_o\dot{C} \rvert$	749	-419	35	749	+14	2
Case	Tactical Diameter	Δ T.D.	% Δ	Time	Δ Time	% Δ
1	1360	--	--	326	--	--
1 (h=10)	1350	-10	0.7	328	+ 2	0.6
2	1375	+15	1	314	-12	4
2 (h=10)	1340	-20	2	307	-19	3
3	1380	+20	2	357	+31	10
4	1250	-110	8	357	+31	10
$R = S_f / \lvert {}_o\dot{C} \rvert$	1234	-126	9	281	-45	14
$R = {}_oS / \lvert {}_o\dot{C} \rvert$	1498	+138	10	281	-45	14

4.2.2.3 Aircraft

Figure C-21 in Appendix C shows course change and bank angle as a function of time. Figure C-22 shows displacement perpendicular to the original path versus displacement in the original direction, called respectively transfer and advance. In constructing the equations which generated these curves, the assumption was made that the aircraft can and does execute a perfectly coordinated turn. In other words, during every instant of the turn the equation $\frac{\omega v}{g} = \tan \phi$ will hold. Once this is true, the resulting turn will always look the same.

4.2.3 Climb/Dive

4.2.3.1 Submarine

There are two curves for submarine dive maneuvers. The first, in Figure C-8 in Appendix C, uses all four cases. It also shows tactical trial data and points generated by the dynamic differential equations. The curves for Cases 1, 2 and 3 for dive angle are very close to one another. There is about a ten percent difference between the kinematic and dynamic results. This is about the same as, or less than, the difference between the dynamic and actual trial data results.

The curves for depth in Cases 1, 2 and 3 are very close to one another. The curve for Case 4 is also close, after the initial time delay. All four show greater depth than the tactical data. This reflects the fact that depth does not really equal the integral of S sin D, but has an initial delay due to an effect similar to side-slip in a turn. The dynamic data seems closer to the tactical data than does the kinematic, only because the dynamic dive angle was smaller than it should have been.

Figure C-9 presents an extended dive maneuver. Only Cases 1 and 2 were presented because of the constant variation of δ_s. As can be seen, both Cases 1 and 2, for 2.5 seconds as well as for 1-second iteration intervals, follow the tactical dive angle quite closely. There is the same phase difference in depth as in Figure C-8, but again the shape of the curve is quite good.

The divergence of depth for both cases, when h = 2.5, is due to the small error in final dive angle.

4.2.3.2 Aircraft

The equations for aircraft climb were not used to generate any response curves.

Very little can be said about computer runs of this maneuver, as there is no basis for comparison.

When the nature of the input, output, and accuracy of each of the four kinematic simulation models is known, the question of cost still remains. In this case, cost is measured in terms of storage space and running time. As described in section 3.6, running time is given in terms of four basic instruction types rather than in terms of actual time in seconds. These four are Transfer Operations, Addition, Multiplication, and Division. Running time and storage are given for two phases of the operation, Initialization and Each Iteration. Storage requirements for initialization includes the space taken up by all constants and input variables used in the maneuver.

Cases 3 and 4 usually include a test of time or of the variables at each iteration. In many cases once the test has been passed it need no longer be performed. The amount of computation required is usually greater while the test is being run than otherwise. In such cases, the maximum number of instructions per iteration is the one given below, in the tables which follow.

The timing and storage necessary for square root subroutines is not given. Those initialization routines which require a square root to be taken have that fact indicated.

Requirements for position updating are listed separately in section 4.3.3

4.3.1 Submarine

4.3.1.1 Acceleration/Deceleration

TABLE 4-10. COMPUTER REQUIREMENTS, SUBMARINE ACCELERATION/DECELERATION

	Transfer	Add	Multiply	Divide	Total Storage	Temporary Storage
Cases 1 and 2						
Initialization	4	1	1	0	11	0
Each Iteration	5	3	3	0	13	1
Case 3						
Initialization	5	3	4	1	19	0
Each Iteration	3	2	1	0	8	0
Case 4						
Initialization	4	2	2	1	14	1
Each Iteration	3	1	0	0	6	0

4.3.1.2 Turn

TABLE 4-11. COMPUTER REQUIREMENTS, SUBMARINE TURN

	Transfer	Add	Multiply	Divide	Total Storage	Temporary Storage
Case 1						
Initialization						
Each Vehicle	8	10	8	2	41	1
*Each Maneuver	12	14	3	1	33	2
Each Iteration						
S	2	5	4	0	12	1
$\dot{C}$	2	4	4	0	10	1
C	1	3	1	0	5	0
Case 2						
Initialization						
Each Vehicle	4	9	6	3	35	1
*Each Maneuver	21	28	15	3	77	2
Each Iteration						
S	2	7	3	0	12	1
$\dot{C}$	3	6	2	0	11	1
C	1	3	1	0	5	0
Case 3						
Initialization						
Each Vehicle	4	9	6	3	35	1
*Each Maneuver	24	29	17	5	80	6
Each Iteration						
S	3	6	0	0	9	0
$\dot{C}$	3	6	0	0	9	0
C	1	3	1	0	5	0
Case 4						
Initialization						
Each Vehicle	4	9	6	3	35	1
*Each Maneuver	22	25	15	2	69	6
Each Iteration						
S	2	4	0	0	6	0
$\dot{C}$	2	4	0	0	6	0
C	1	1	0	0	3	0

*This does not include square root subroutine.

TABLE 4-12. COMPUTER REQUIREMENTS, SUBMARINE DIVE

	Transfer	Add	Multiply	Divide	Total Storage	Temporary Storage
Case 1						
Initialization	8	9	7	0	34	1
Each Iteration	3	7	3	0	52	6
Case 2						
Initialization	10	9	7	1	41	1
Each Iteration	3	7	2	0	50	5
Case 3						
Initialization	5	7	3	2	28	2
Each Iteration	6	3	1	0	44	5
Case 4						
Initialization						
*D_0, $\dot{D}_0 \neq 0$	31	9	5	6	61	12
**D_0, $\dot{D}_0 = 0$	13	3	4	3	30	4
Each Iteration	4	2	1	0	6	0

*This does not include the use of a cosine subroutine three times and a sine subroutine twice.

**This does not include the use of one sine subroutine and one cosine subroutine.

4.3.2 Surface Vessel

4.3.2.1 Acceleration/Deceleration

TABLE 4-13. COMPUTER REQUIREMENTS, SURFACE VESSEL ACCELERATION/DECELERATION

	Transfer	Add	Multiply	Divide	Total Storage	Temporary Storage
Cases 1 & 2						
Initialization	3	0	3	0	10	1
Each Iteration	2	3	2	0	8	0
Case 3						
Initialization	10	4	4	1	25	2
Each Iteration	4	2	1	0	8	0
Case 4						
Initialization	5	2	4	1	17	2
Each Iteration	2	1	0	0	5	0

TABLE 4-14. COMPUTER REQUIREMENTS, SURFACE VESSEL TURN

	Transfer	Add	Multiply	Divide	Total Storage	Temporary Storage
Case 1						
Initialization						
Each Ship	12	13	14	1	62	1
Each Maneuver	7	7	1	0	24	0
Each Iteration						
C	10	17	11	0	39	2
V	2	6	5	0	13	1
Case 2						
Initialization						
Each Ship	12	12	14	1	53	1
***Each Maneuver**	13	13	10	2	51	3
Each Iteration						
C	11	19	10	0	40	2
V	2	6	5	0	13	1
Case 3						
Initialization						
Each Ship	9	15	10	3	50	2
***Each Maneuver**	11	13	15	2	51	2
****Each Iteration**						
C	10	14	1	0	31	2
V	3	4	0	0	12	0
Case 4						
Initialization						
Each Ship	9	15	9	3	48	2
***Each Maneuver**	13	16	13	3	69	2
Each Iteration						
C	4	4	0	0	5	0
V	2	3	0	0	5	0

*Does not include square root subroutine.

**The first time $\dot{C}_n = {}_o\dot{C}$ and the first time $V_n = V_f$, there are two more transfer operations and two more addition operations.

4.3.3 Position Updating

When both X and Y are updated, the requirements are:

TABLE 4-15. COMPUTER REQUIREMENTS, POSITION UPDATING

	Transfer	Add	Multiply	Divide	Total Storage	Temporary Storage
X, Y						
Initialization	4	4	0	0	10	2
Each Iteration	11	15	9	0	40	5
When only one is updated, as may occur in dive maneuvers, they are:						
Each Iteration	9	13	10	0	38	5

4.3.4 Aircraft

The figures for aircraft computer requirements are based on the storage and time used by the Sylvania 9400 General Purpose digital computer. The requirements for the algorithms used to derive the inputs to the vehicle simulator were computed for the aircraft turn and climb. The entire program for these maneuvers is shown in the flow charts of Appendix D.

4.3.4.1 Acceleration

Initialization with respect to altitude, ${}_{o}S$, and h:

Storage, 12 locations
Time, 0.208 milliseconds

Each loop:

Storage, 23 locations
Time, 0.456 milliseconds

4.3.4.2 Coordinated Turn

Approximate storage and timing requirements

Initialization for ${}_{o}\phi$ and $(\Delta C)_f$:
Time, 1.0 milliseconds

Each loop while rolling into bank angle:
Time, 2.3 milliseconds

Each loop at constant bank angle:
Time, 2.2 milliseconds

Initialization loop to roll out of bank angle:
Time, 3.2 milliseconds

Each loop while rolling out of bank angle:
Time, 3.0 milliseconds

Total Storage: 400 locations

Computation of ground coordinates x and y required 1.9 milliseconds in each loop.

4.3.4.3 Climb

This involves approximate storage and timing requirements for a program forereaching ordered speed and either ordered altitude or ordered heading, starting with initial speed, altitude, and pitch angle. There are two alternative forms of this program, using simple or more complicated equations. Requirements are given for each form.

These requirements correspond to the program whose flow chart is in Appendix D, Figure D-2. The program was never run; these figures are only estimates.

Simple equations:
Storage, 285 locations

Initialization for h, $_0v$, v_0, θ_0, initial altitude, and $_0\gamma$:
Time, 1.3 milliseconds

Each loop:
Time, 2.6 milliseconds

Initialization for h, $_0v$, v_0, θ_0, initial altitude, and ordered altitude:
Time, 1.6 milliseconds

Each loop:
Time, 2.6 milliseconds

More complete equations:
Storage, 309 locations

Initialization for h, $_0v$, v_0, θ_0, initial altitude, and $_0\gamma$:
Time, 1.5 milliseconds

Each loop:
Time, 2.8 milliseconds

Initialization for h, $_0v$, v_0, θ_0, initial altitude, and ordered altitude:
Time, 1.8 milliseconds

Each loop:
Time, 2.9 milliseconds

SECTION V

DISCUSSION OF RESULTS

5.1 EVALUATION OF ACCURACY

Section 4.2 contains tables listing the results of the trial runs of the kinematic equations. These tables correspond to the response curves in Appendix C, and give percent error for each kinematic model compared against a standard. In some cases the standard is the response curve generated by the simplified dynamic equations listed in sections 3.3.1 and 3.3.2. In other cases it is the response curve generated by the Case 1 kinematic equation, using a one-second iteration interval. For aircraft acceleration, the standard is approved Performance Data (Reference 3).

The percent errors are discussed below. Models are called "very good," "good," "fair," "poor," or "unacceptable" depending on their percent error. Since the simplified dynamic data was used as a basis, its accuracy must be taken into account when discussing the accuracy of the other models.

Evaluation is based on direction of error as well as size, although no rule is formulated for doing this. Simulation models are called "very good" if their percent errors are all less than 7 percent, "good" if they are all between 5 percent and 17 percent. They are called "fair" if they are between 14 percent and 27 percent, and "poor" if they are all between 24 percent and 50 percent. Models with larger error percentages are unacceptable. The overlap is to allow for a certain amount of qualitative evaluation.

5.1.1 Acceleration/Deceleration

Cases 1 and 2 are very good for both submarine and surface vessels. Case 3 varies. For the submarine, it is very good for acceleration and speed loss in the turn, but poor for deceleration. For the surface vessel, it is very good for speed loss in a turn as long as $V_o - V_f$ is small. When $V_o - V_f$ grows, Case 3 gives too small a distance. The error is small, but since it is always biased in the same direction, Case 3 will be called "good" rather than "very good."

No examples are shown for surface vessel acceleration/deceleration. On the basis of the results shown for speed loss in a turn, and the arguments below in section 5.3.2.1, it is assumed that Case 3 for this case will give very good results.

The kinematic model for aircraft acceleration can be used with confidence only at speeds between 250 knots and 600 knots, and only at altitudes below 15,000 feet. To use the acceleration formulas at higher altitudes entails greater limitations on the speeds in order to have any accuracy at all. In order to reduce errors to the order of 10 percent rather than 50 percent, it is necessary to have a more complicated formula or a table look up for b and c. (See section A.3.1 of Appendix A.)

5.1.2 Turn

5.1.2.1 Submarine

Table 4-5 in section 4.2.2.1 shows the percent errors in advance, transfer, tactical diameter, and time to turn 180 degrees for the various cases of the submarine kinematic equations. For each case, there are four separate percent errors. The greatest error is in advance; except for advance, Cases 1 and 2 are very good while Cases 3 and 4 are good. This statement is true despite the high error in time-to-turn for Cases 3 and 4, since this error and the error in advance are both due to the too high initial time delay. This is further confirmed by the accuracy of Case 2, using the ten-second iteration interval. In that run, the two-second (T/2) initial time delay was neglected.

Table 4-6 reinforces this analysis even further. Transfer and tactical diameter are good for nine turns using Case 4. Advance is poor, and time-to-turn is fair. Therefore, until the time delay is adjusted, Cases 3 and 4 will be called "poor" and Cases 2 and 1 "good."

The two radii referred to in Table 4-5 are examples of the original kinematic methods, provided as an alternative to Case 4. The circle with $S = S_f$ gives better overall results than does Case 4. This shows that the time delays in Cases 3 and 4 are so erroneous that no time delay at all would give better results.

Table 4-6 compares advance, transfer, tactical diameter, and time to turn 360 degrees for nine submarine turns for the kinematic equations and the tactical trial data; $\blacktriangle$ and % $\blacktriangle$ are with respect to the dynamic equations. Using the dynamic data as a standard, the difference between the dynamic and tactical trial data yields a percent error as high as 55 percent in one instance. Of the thirty-six pairs of values compared, however, two show errors greater than 30 percent, three show errors greater than 20 percent and eleven had errors greater than 10 percent. Furthermore, these errors are not all in the same direction, but are rather evenly divided between positive and negative errors. The data analyzed in Table 4-6 appears in Tables C-I, C-II and C-III in Appendix C.

5.1.2.2 Surface Vessel

Tables 4-8 and 4-9 show percent error figures for the two worst cases of surface vessel turn. These were determined by inspection of the results shown in Table 4-7. (Refer to discussion on page 4-10). Case 1, with a one-second iteration interval, is used as a standard on the basis of its accuracy in Figure C-10 of Appendix C. From the figures in the cited tables, Cases 2, 3, and 4 can all be called good; they all give better results than the circles. When Case 1 is used with a 10-second iteration interval, it gives very good results. Case 2 with a 10-second iteration rate produces results as good as Case 3.

5.1.2.3 Aircraft

The kinematic equation for an aircraft turn is accurate as long as the assumption remains valid that the turn is perfectly coordinated.

5.1.3 Dive/Climb

There are no tables for the accuracy of the kinematic equations for submarine dive. Inspection of Figures C-8 and C-9 shows that all four cases give good results. Only one of these figures shows Cases 3 and 4, however, and this is a very special sort of maneuver. The inflexibility of Cases 3 and 4 will sometimes lead to results that are only fair; at other times the results will be good.

There was nothing available with which to compare the kinematic model for aircraft climb.

5.1.4 Summary of Evaluation

The following tables summarize the evaluation of the kinematic models in section 5.1, the evaluations assuming an iteration interval of one second. Not enough was said about the aircraft to warrant further mention here. The explanation of the categories is in the introductory paragraphs of section 5.1. They correspond to the following percent errors.

very good	0% - 7%
good	5% - 17%
fair	14% - 27%
poor	24% - 50%

TABLE 5-1. EVALUATION OF SUBMARINE KINEMATIC MODELS

Maneuver	Case 1	Case 2	Case 3	Case 4
Acceleration	very good	very good	very good	very good
Deceleration	very good	very good	poor	poor
Speed Loss in Turn	very good	very good	very good	very good
Turn	good	good	poor	poor
Dive	good	good	good/fair	fair

TABLE 5-2. EVALUATION OF SURFACE VESSEL KINEMATIC MODELS

Maneuver	Case 1	Case 2	Case 3	Case 4
Acceleration/ Deceleration	very good	very good	very good	very good
Speed Loss in Turn	very good	very good	good	good
Turn	very good	good	good	good

5.2 ITERATION INTERVAL

The equations under discussion are all constructed for solution on a digital computer. The kinematic equations are almost all difference equations, and the dynamic equations must be solved using difference equations to update the variables. The size of the iteration intervals involved in these difference equations can have a considerable effect upon the accuracy of the simulation.

The dynamic equations used as a basis for the derivation of the submarine and surface vessel kinematic equations are solved on the digital computer by using open, multi-step difference equations. One property of this solution technique is that, for each set of differential equations, there is an upper limit for the values which can be used for h, the iteration interval. If h exceeds this maximum, the solution begins to diverge.

Kinematic equations respond somewhat differently to the size of h. In Cases 1 and 2, the validity of the exponential expansion depends on the size of h. The error can grow quite large, but never diverge. In Cases 3 and 4, the error has an upper bound which grows linearly with h.

5.2.1 Cases 1 and 2

The results displayed in Appendix C provide insufficient information for a systematic analysis of the effects of increasing the size of the iteration interval. Iteration intervals larger than one second were used for several maneuvers with very good results. From these results we can conclude that there is some flexibility with respect to iteration interval, but we cannot say how much.

Appendix B discusses the restrictions associated with each kinematic model. These restrictions consist of upper bounds on the size of the product of speed and iteration interval. For those submarine maneuvers which were run with $h > 1$, Sh was always less than the appropriate upper limit. In the case of the surface vessel, however, Vh was always greater than this upper limit. In the DD-445 runs, Vh is 2 to 3 times the upper limit; in the CA-68 runs it is 15 to 30 percent higher. The good results in all these cases indicate that there is some room for re-examination of the expressed upper bounds.

5.2.2 Case 4

The larger values of h used in the test runs in Appendix C were used only for Cases 1 and 2. For Cases 3 and 4 there is no upper limit on the size of the iteration interval. Instead of the error diverging as the iteration interval exceeds some fixed value, the error is directly proportional to the size of the iteration interval. Consider, for example, Case 4 used to find $\dot{C}$ where $\dot{C}_o = 0$.

At time $t = 3h$, the true value of C is the area under the lines connecting the points $(0,0)$, $(T_c, 0)$, $(T_c, {}_o\dot{C})$ and $(3h, {}_o\dot{C})$. This area is ${}_o\dot{C}(3h - T_c)$.

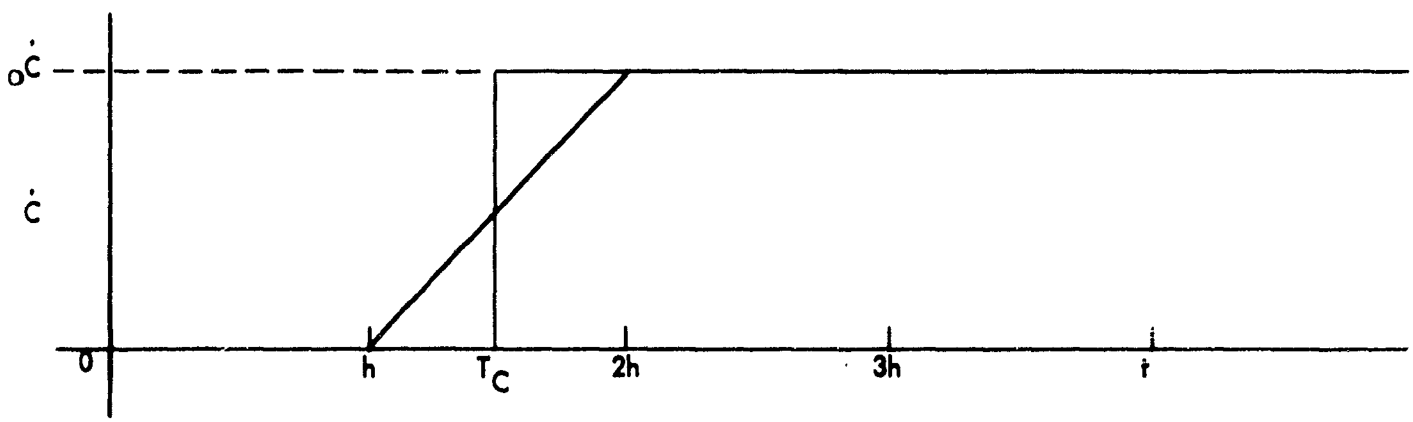

Figure 5-1. Case 4; Error as Function of h

Sampling every h seconds, $C_{n_s} = C_{n-1_s} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1})$, where C_n = C at time $t = nh$. When $t = 3h$, $C_{3_s} = \frac{3}{2} h\, {}_o\dot{C}$. The difference between C_{3_s} and the true value at $t = 3h$ is $\epsilon = {}_o\dot{C}(\frac{3}{2} h - T_c)$. Since $h < T_c < 2h$, the error has upper and lower bounds ~~such~~ that $-\frac{{}_o\dot{C}h}{2} < \epsilon < \frac{{}_o\dot{C}h}{2}$ Thus $|\epsilon| < {}_o\dot{C}h/2$. As h grows, the error in C is always less than ${}_o\dot{C}h/$

5.2.3 Case 3

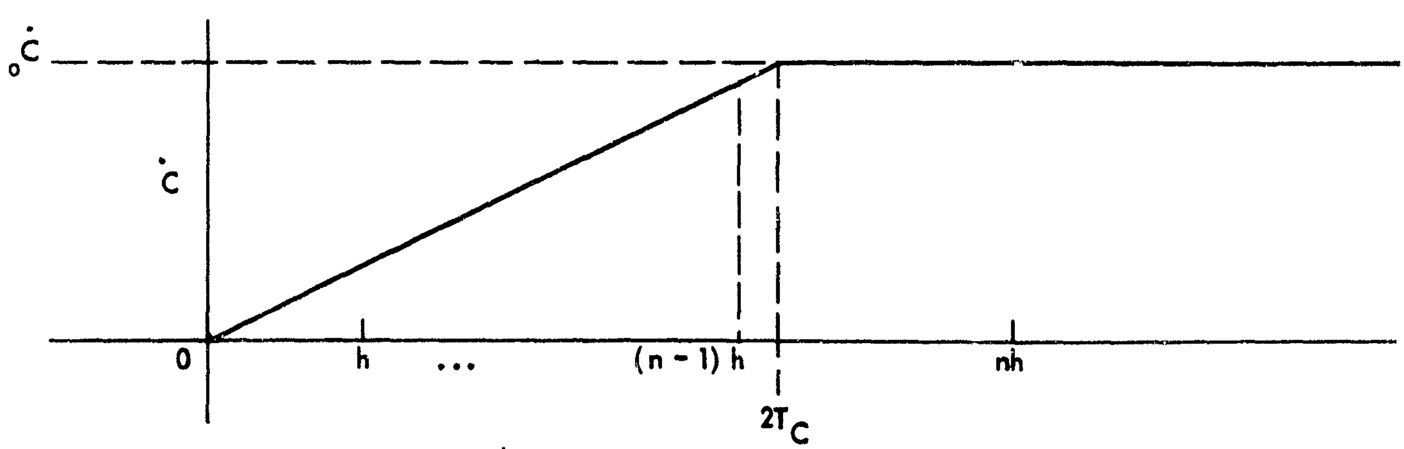

Figure 5-2. Case 3; Error as Function of h

The choice of $2T_C$ for the time delay to reach $o\dot{C}$ is due to the delay for Case 3 being twice that for Case 4.

The error for Case 3 is even smaller. The true value of C at $t = nh$ is the area under the $\dot{C}$ curve, $C_{n_t} = {}_o\dot{C}(\frac{2T_c}{2} + nh - 2T_c) = {}_o\dot{C}(nh - T_c)$. Sampling every h seconds, $C_{n_s} = C_{n-1_s} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1})$. At time $t = (n-1)h$, $\dot{C}_{n-1} = \frac{{}_o\dot{C}(n-1)h}{2T_c}$. $C_{n-1_s} = \frac{{}_o\dot{C}(n-1)^2h^2}{4T_c}$

At time $t = nh$,

$$C_{n_s} = \frac{{}_o\dot{C}(n-1)^2h^2}{4T_c} + \frac{h}{2}\left({}_o\dot{C} + \frac{(n-1)\,{}_o\dot{C}h}{2T_c}\right)$$

The difference is $\epsilon = {}_o\dot{C}\left[(n - \frac{1}{2})h - T_c - \frac{h^2}{4T_c}\left((n-1) + (n-1)^2\right)\right]$. In this example, $(n-1)h < 2T_c < nh$. When $2T_c$ equals $(n-1)h$ or nh, $\epsilon = 0$. The maximum value for $|\epsilon|$ is at $2T_c = h\sqrt{n^2 - n}$. That maximum is ${}_o\dot{C}h(n - \frac{1}{2} - \sqrt{n^2 - n})$ which equals ${}_o\dot{C}h/2$ when $n = 1$, $0.086\,{}_o\dot{C}h$ when $n = 2$, $0.051\,{}_o\dot{C}h$ when $n = 3$, and continues to decrease as n grows larger.

The upper bound on ϵ is easier to see when expressed in terms of T_c. Since the maximum is at $2T_c = h\sqrt{n^2 - n}$, $n = \frac{1}{2}(1 + \sqrt{1 + 16T_c^2/h^2})$ and the maximum is $\left(\sqrt{(2T_c)^2 + \frac{h^2}{4}} - 2T_c\right){}_o\dot{C}$. If $2T_c < h/4$, then $n = 1$ since $(n-1)h < 2T_c < nh$. The maximum is ${}_o\dot{C}/2$. If $n > 1$, than $\frac{h^2}{4(2T_c)^2} < 1/4$ and the radical can be expanded. This gives $|\epsilon| < \min\left(\frac{{}_o\dot{C}h}{2}, \frac{{}_o\dot{C}h^2}{16T_c}\right)$.

In general, for any variable $\dot{y}$ the error in y is always less than $|\dot{y}_f - \dot{y}_o|h/2$ for Case 4 and $|\dot{y}_f - \dot{y}_o|h/20$ for Case 3 (assuming $n \geq 3$).

5.3 SOURCES OF ERROR

Appendix A contains the derivations of the kinematic equations. There are many places in these derivations where simplifying assumptions or approximations are made. The justification for these methods lies, for the most part, in their effectiveness. Not all the results were equally good, however. In the following paragraphs, those parts of the derivations will be pointed out where improvement is needed.

The discussions of the submarine or surface vessel are concerned with the integral used to derive the equations for Cases 3 and 4 (equations 3-1 and 3-2), since Cases 1 and 2 were always either good or very good (Tables 5-1 and 5-2). The discussion of the aircraft model is more general.

5.3.1 Submarine

5.3.1.1 Deceleration

The evaluation of the integral of S for the instructor's control appears in section A.1.2.2 of Appendix A. The expression for m_a, the slope of the S versus time curve in the instructor control equation, is shown in equation 35 in Appendix A. This equation is derived from the expression

$$m_a = \frac{1}{2}({}_oS - S_o)^2\left[\frac{1}{A_1(1 + A_3)}\log\left(1 + \frac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o}\right)\right]^{-1} \tag{5-1}$$

Equation 35 is found from this equation by approximating the logarithm by the first term of its series expansion.

$$\text{i.e.,}\ \log\left(1 + \frac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o}\right) \approx \frac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o} \tag{5-2}$$

The validity of equation 5-2 depends very much on the size of its right-hand member. Since $1 + A_3 = 0.25$, an acceleration from S_o to ${}_oS$ where $S_o < {}_oS$ will be better simulated by equation 5-1 than a deceleration where ${}_oS < S_o$. Consider, as an example, the maneuver shown in Figure C-2 in Appendix C. That is a deceleration from 20 knots to 2 knots. Since

$A_3 = -0.75$, $\dfrac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o} = -0.643$ but $\log(1 - 0.643) = -1.030$.

$$\text{Percent error} = \frac{1.030 - 0.643}{1.03} = 38\%$$

whereas, for an acceleration from 2 knots to 20 knots,

$\dfrac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o} = 0.220$ and $\log(1 + 0.220) = 0.199.$

$$\text{Percent error} = \frac{0.220 - 0.199}{0.199} = 11\%$$

which is a considerable improvement.

The foregoing discussion indicates the limitations of Cases 3 and 4 of the speed equations, especially for deceleration. Improvement could be effected by using a better approximation for the logarithm function. In the above example, the use of -1.03 instead of -0.643 in the deceleration maneuver results in a distance covered of 2,638 yards in 1200 seconds for a percent error of 0.65 (Table 4-2).

5.3.1.2 Advance

The results listed in Tables 4-5 and 4-6 in section 4.2.2.1 demonstrate the inadequacy of Cases 3 and 4 of the kinematic formulation for submarine turn. The error is in the formula for T_c, the time delay for $\dot{C}$. By comparing the results using a circle with $R = S_f/|{}_0\dot{C}|$ and no time delay with Cases 3 and 4 in Table 4-5, we see that the original kinematic formulas are better than those using the present derived value of T_c as time delay.

This error is not due to the approximation of the exponential. The discussion leading up to equation 46 in Appendix A contains the equation

$$\frac{1}{{}_0S}\exp\left(\frac{A_{6o}S\tau_1\delta_r}{F_c}\right) - \frac{1}{S_f}\exp\left(\frac{A_6S_f\tau_1\delta_r}{F_c}\right) = \left(\frac{1}{{}_0S} - \frac{1}{S_f}\right) + \frac{A_6^2\tau_1^2\delta_r^2}{2F_c^2}\left({}_0S - S_f\right).$$

This equation is the sole approximation of the exponential used in the derivation of T_c. A check of this equation, for the SS(B)N598 entering a 20-degrees rudder angle turn at 10 knots yields -0.084 for the difference in the exponentials and -0.081 for the approximation. This is well within the range of desired accuracy, whereas the error in advance for that same maneuver was 67 percent.

The source of the error is hidden somewhere else in the derivation and has not yet been found.

5.3.2 Surface Vessel

5.3.2.1 Speed Loss in Turn

Cases 3 and 4 of the kinematic model for speed loss in a submarine turn generate too rapid a drop in speed. This is due to an approximation made in the evaluation of the integral (section A.2.2.2.2, Appendix A). Equation 137, used in that evaluation, is $\dot{V} = a_7a_{6o}\dot{C}^2 + a_8(V_o^2 - V^2)$. The term $a_7a_{6o}^2\dot{C}^2$ is a constant used to approximate $a_7(a_6y - \dot{\alpha})$ Actually $a_{6o}^2\dot{C}$ is an upper bound for $(a_6y - \dot{\alpha})^2$, which the latter approaches asymptotically.

a_7 is negative and $a_8(V_0^2 - V^2)$ is positive but smaller in absolute value than $-a_7 a_{60}^2 \dot{C}^2$. As V decreases, $a_8(V_0^2 - V^2)$ grows and $\dot{V}$ goes to zero. It is the ${}_0\dot{C}^2$ term then, which causes the negative value of $\dot{V}$.

Since this has larger absolute value than would be the case if $(a_6 y - \dot{\alpha})^2$ were used, $-\dot{V}$ is too large. Therefore, the surface vessel slows down too fast.

No other constant term could be used in place of $a_7 a_{60} \dot{C}^2$ without changing the final value of V. The only alternative is to use the Case 3 equation for $\dot{C}$, letting $y = \dot{C}$ and ignoring $\dot{\alpha}$. This should improve the output somewhat, provided the resulting integral can be evaluated.

5.3.2.2 Turn, Case 2

The improvement of the equation for Case 2 of the surface vessel turn was accidental Response curves were generated using equations 121 and 122 of Appendix A. In almost all cases, C increased too quickly. For one surface vessel, it did not increase quickly enough. It was evident that the equations used for Case 1 gave a very good fit, however, so the Case 2 equations were changed in such a way as to resemble the Case 1 equations more closely. This entailed replacing ${}_0\dot{C}$ by $V_{n-1}\,{}_0\dot{C}/V_f$, which is equal to $V_{n-1}\xi\delta$. In the response curves generated by these revised equations, C increased even faster than before, as was expected. It was noticed, however, that by changing the initial time delay from T/2 to T, the error was considerably diminished. This latter form was therefore adopted.

5.3.3 Aircraft

5.3.3.1 Speed

A statement is made in section 5.' 1 regarding the loss of accuracy with increasing altitude of the kinematic model for the speed of an aircraft. This is due to two things. The first is the inaccuracy of the linear approximations of b and c as functions of altitude (equations 156 and 157 of Appendix A, and the table of b and c values preceding them). The second is the deviation from the hyperbolic shape that the curve of thrust versus velocity undergoes at high altitudes. This can be seen in the curves in Reference 2. The two effects combine to give errors near 50 percent. Despite this very large error, the model is still acceptable. This is because the model is very simple and the curves being fitted are extremely complex.

5.3.3.2 Climb

Of the two formulas used in the kinematic model for aircraft climb, the formula for air speed as a function of attitude angle is the more sensitive one. For higher airspeeds or higher altitudes the linear approximation no longer holds. The form of the function would the

be a series of connected line segments with a table look-up necessary to find the end points of the line segments as a function of level-flight air speed and altitude.

The quantity k which appears in the formula for heading as a function of attitude angle is proportional to the density of the air, ρ, which in turn is a function of altitude. This dependence can be approximated by either an exponential in altitude or by a series of line segments with a table look-up.

5.4 COMPUTER LOAD

Section 4.3 contains tables giving timing and storage requirements for the various kinematic models. An examination of the timing requirements shows that there are two areas that need further explanation. Each of the kinematic models requires some initialization computation as well as the computation at each iteration. In actual use, the time per iteration allotted to each maneuver model has to be very nearly constant. The high initialization time of most models might raise this constant running time to allow time for initialization whenever it is necessary. This possibility is discussed in section 5.4.1.

Another area that needs clarification is the comparative time requirements for the four cases of each maneuver. The tables in section 4.3 give time requirements in terms of four basic instructions. This kind of tabulation does not lend itself to easy comparisons. Section 5.4.2 includes comparisons between the models based on typical fixed- and floating-point instruction times.

5.4.1 Peak Load at Initialization

If limited computation time is available for each maneuver, the time required for initialization may prove a burden. There are several ways of compensating for this peak at initialization.

5.4.1.1 Case 1

In the submarine turn and submarine dive maneuvers, the value of the control plane angle appearing in the equation for $\dot{C}_n$ and D_n respectively is the value at time $h(n - 1)$. This means that if the control plane angle is zero at time $t = 0$, no change will occur in the variabl until $t = 2h$. This leaves two iterations for initialization.

This situation is also true to some extent for the surface vessel turn. There, the only term that has any effect at $t = h$ is $V_{n-1}ha_5r_2\delta_1$ (see equation 46, Appendix B). This leaves considerable time for initialization at the zeroth and first iteration.

5.4.1.2 Cases 2 and 4

When using Case 2 or Case 4 (Case 3 or Case 4 in a submarine dive maneuver), ther is an initial time delay. This time delay is always on the order of several seconds. Countii can start with the initial order, with complete confidence that initialization will be complete before the time delay is used up.

5.4.1.3 Case 3

A certain amount of running time per iteration in Case 3 is used to test whether or not the variable has reached steady-state value. In the arithmetic portion of each iteration, one addition is used to update the variable by the constant increment, while two additions an a multiplication are used to update the integral of the variable, if necessary.

The test and most of the arithmatic can be eliminated at the first iteration. The test is unnecessary at the first iteration and the constant difference itself is the first value of the variable, while half of it is the first value of the integral of the variable. Thus, by perform ing half the initialization at nh = 0 and the other half at nh = h, the peak at initialization can be considerably reduced for Case 3.

5.4.1.4 Turn Maneuvers

For turn maneuvers in general, where the position of the vehicle has to be updated, there is another place where time can be saved for initialization. A glance at the curves of x versus y for turn maneuvers shows that several seconds elapse before y becomes notic ably different from zero. The time saved by neglecting to update y during the first few iter tions can be used for initialization.

5.4.2 Comparative Time Requirements

The following ratios have been selected as typical of modern high-speed digital computers.

	Transfer	Add	Multiply	Divide
Fixed Point	1	1	2	5
Floating Point	1	4	3	6

5.4.2.1 Submarine

TABLE 5-3. COMPARISON OF TIME REQUIREMENTS: SUBMARINE

	Case 1		Case 2		Case 3		Case 4	
	Fx	Fℓ	Fx	Fℓ	Fx	Fℓ	Fx	Fℓ
Acceleration/Deceleration								
Initialization	7	11	7	11	21	35	15	24
Each Iteration	14	26	14	26	7	14	4	7
Turn								
Initialization								
Each Vehicle	44	84	40	76	40	76	40	76
*Each Maneuver	37	83	94	196	112	221	87	179
Each Iteration								
S	15	34	15	22	7	14	6	18
$\dot{C}$	14	30	13	23	9	27	6	18
C	6	16	6	16	6	16	2	5
Total	35	80	34	61	22	57	14	41
Dive								
**Initialization	31	65	38	73	28	54	40	118
***Each Iteration	16	40	14	37	11	21	8	15

*Numbers do not include square root subroutine used to find $_0\dot{C}$ in Case 1 and S_f in Cases 2, 3, and 4.

**Numbers are for the general case where D_0 and $\dot{D}_0$ are not zero; they do not include two sine routines and three cosine subroutines.

***Numbers are for computation of D. To update Z, add 42 fixed units and 91 floating units to time per iteration of Cases 1, 2 and 3 and 3 fixed units and 6 floating units to the iteration time of Case 4.

Summing up, the ratios for time per iteration are roughly as follows:

	Fixed	Floating	
A/D	4:4:2:1	4:4:2:1	(Case 1, Case 2, Case 3, Case 4)
Turn	3:3:2:1	4:3:3:2	
Dive	4:4:3:2	2:2:1:1	

These ratios are for the outputs speed, course angle and dive angle. To get position, add the following for Cases 1, 2, and 3.

A/D : 5 fixed, 12 floating

Turn : 8 fixed, 20 floating for initialization

44 fixed, 98 floating each iteration

Dive : 42 fixed, 91 floating

For Case 4, A/D and Dive are different.

A/D : 3 fixed, 6 floating

Dive : 3 fixed, 6 floating

The new ratios are

		Fixed	Floating
A/D	:	3:3:2:1	3:3:2:1
Turn	:	7:7:6:5	9:8:8:7
Dive	:	5:5:5:1	6:6:5:1

5.4.2.2 Surface Vessel

TABLE 5-4. COMPARISON OF TIME REQUIREMENTS, SURFACE VESSEL

	Case 1		Case 2		Case 3		Case 4	
	Fx	Fℓ	Fx	Fℓ	Fx	Fℓ	Fx	Fℓ
Acceleration/Deceleration								
Initialization	9	12	9	12	27	44	20	31
Each Iteration	9	20	9	20	3	15	3	6
Turn								
Initialization								
Each Ship	58	112	54	108	59	117	57	114
*Each Maneuver	16	38	56	107	64	120	70	134
Each Iteration								
C	49	111	50	117	26	69	8	20
V	18	41	18	41	7	19	5	14
Total	67	152	68	158	33	88	13	34

*Numbers do not include square root subroutine used to find V_f.

The ratios for time per iteration for speed and course angle are

	Fixed	Floating
A/D	3: 3:3:1	4:4:3:1
Turn	10:10:5:2	5:5:3:1

After adding 5 fixed units and 12 floating units to Cases 1, 2, and 3 of A/D, 3 fixed units and 6 floating units to Case 4 of A/D and 44 fixed units and 98 floating units to all cases of turn, the ratios for position are roughly as follows:

	Fixed	Floating
A/D	2: 2:2:1	8:8:7:3
Turn	10:10:7:5	4:4:3:2

SECTION VI

CONCLUSIONS

6.1 EXTENT OF DEVELOPMENT OF A THIRD CATEGORY OF VEHICLE SIMULATION MODELS

The third category of vehicle simulation models is defined in Section II as falling between the two that already exist, with respect to both fidelity and computer requirements. The two already existing are the dynamic and kinematic models defined in sections 1.1.1 and 1.1.2. By comparing the new kinematic equations in Appendix B with the dynamic equations in section 3.3, it can be seen that the new models fall between these two, with respect to computer requirements. Each of the dynamic equations in section 3.3 must be used with a numerical integration formula, at each iteration; the resulting running time per iteration is about 50 percent more than that of Case 1 of the corresponding kinematic model.

The dynamic equations for aircraft motion are not shown, but in their simplest form they are much more complicated than the Appendix B kinematic equations for the aircraft.

It is not required that the new kinematic models place more of a burden on the computer than the old kinematic models, nor need they be less accurate than the simple dynamic models. It is sufficient that they burden the computer less than do the simple dynamic models, while being more accurate than the kinematic models now in use. As can be seen from Tables 4-5, 4-8 and 4-9, they are indeed more accurate than the old kinematic models. In Table 4-5, the old kinematic formula for a turn is more accurate than cases 3 and 4, but less accurate than cases 1 and 2. Similarly, although Cases 3 and 4 of the submarine deceleration equations are inaccurate, Cases 1 and 2 are more accurate than the old kinematic equations (Table 4-2).

The advantage of the new kinematic models over the existing simulation methods is further enhanced by the flexibility of iteration interval. This is especially true in those cases where Case 3 gives good results (see Tables 5-1 and 5-2). There, the error grows very slowly as a linear function of h (see section 5.2.2).

The kinematic model for the aircraft is also an improvement, and better than existing target simulators. Its simplicity means that, with fewer calculations than required by existing target simulators, it will incorporate more aircraft characteristics. In addition to time delays, it supplies the instantaneous angular orientation of the aircraft necessary for simulating radar profiles. It also gives the relation at any instant between air speed, heading and attitude, thereby providing the observer with an indication of the type of aircraft involved.

Furthermore, these important aspects of target simulation are preserved as the iteration interval grows from one-tenth of a second to two or three seconds, a 10- to 50-fold increase over the iteration interval necessary for even gross dynamic aircraft simulation

6.2 ADVANTAGEOUS SIMULATION SITUATIONS FOR MODELS IN THIS CATEGORY

Section 3.1.2 contains a description of the simulation circumstance normally associated with each of the four kinematic models developed for the submarine and surface vessel. The names Operator Control, Command Control, Instructor Control and Program Control are derived from these descriptions. A correspondence is established between use and model, based on the form of input and output and the expected accuracy and computer requirements of each model.

In Section IV, the elements of this basis are re-examined. In section 4.1 it is pointed out that, except in the case of the submarine dive, the input for any model can be adapted to any other model. Furthermore, if position is the variable of primary interest, the outputs from Cases 1, 2 and 3 are very similar. Also, the actual accuracies and computer requirements of each model, listed in sections 4.2 and 4.3, are somewhat differe. from the expected values.

The correspondence between use and model is therefore revised, based on the actual results in Section IV and the discussion of these results in Section V.

6.2.1 Acceleration/Deceleration

For the acceleration/deceleration maneuver, the original correspondence between control situations and equations is a good one. Operator and command control are the own-ship motion situations. They both use the Case 1 equation, which is the same as the Case 2 equation for this maneuver. The Case 3 equation is not good for own-ship motion because of the abrupt, unrealistic change in acceleration when ${}_0S$ is reached.

Target motion is simulated using Case 3 or 4 becaus of the saving in computer time. Case 4 is used only for the program control situation because abrupt velocity change in the output is very evident and too unrealistic for any other situation.

6.2.1.1 Submarine

There are three different computer burdens corresponding to three different shapes of the output from the three submarine acceleration/deceleration models. Therefore the original correspondence between control situations and case numbers is retained. This is the case even for the deceleration, where cases 3 and 4 give results that are poor but still adequate for target simulation.

6.2.1.2 Surface Vessel

The difference in time requirements between Cases 1 and 3 is not enough to justify the use of Case 3; the exception to this is when the simulation system calls for the use of

a large iteration interval. In such a case it would be advantageous to use Case 3 because Cases 1 and 2 may be liable to error or instability as h increases. Otherwise, Case 1 can be used for all situations where the velocity change may not be abrupt, Case 4 for all cases where it may; accuracy is very good in either case.

6.2.1.3 Aircraft

The aircraft acceleration simulation model can be used at low speeds and low altitudes as part of an own-ship model. At higher speeds and altitudes, although too inaccurate for an own-ship simulator, the short running time and flexibility of iteration interval make it an acceptable part of a target simulator.

6.2.2 Turn

In the submarine turn maneuver, the differences in the form of the output are least apparent. This is especially true for Cases 1, 2 and 3, whether used as an own-ship or target simulator. Case 4 has a response that could be detected as different in an own-ship simulator, but not in a target simulator where the only output is position as a function of time. In the surface vessel turn maneuver, the differences in output are more apparent.

6.2.2.1 Submarine

There is little relative difference between the computer running times for Cases 1, 2 and 3, especially when the time to update position is included. Furthermore, Case 3 as it now stands is very inaccurate. Therefore, Case 1 should be used for the operator control situation, Case 2 for the command and instructor control situation, and Case 4 for the program control situation. In Case 4, instead of using the formula for T_c shown in Appendix B, T_c should be set to T, the time for the rudder to move.

In the event that a formula can be devised giving a better value for T_c, then Case 3 should be used for the command control, instructor control, and program control situations doubling the size of the iteration interval for the program control situation.

6.2.2.2 Surface Vessel

For the surface vessel, Case 3 was more consistently accurate than Case 2. Therefore Case 1 should be used for the operator control situation, Case 3 for the command and instructor control situations, and Case 4 for the program control situation. This rule need not be rigidly followed. If the operator control maneuvers are to be simple turns at a constant rudder setting, then Case 3 can be used instead of Case 1. If a large iteration interval is more desirable than the savings per iteration using Case 4, then Case 3 can be used for the program control situation.

6.2.2.3 Aircraft

The aircraft turn model can be used in any situation where there is no need to simulate the deviations from a coordinated turn. This means it can be used in most target and a few own-ship simulators.

6.2.3 Dive/Climb

6.2.3.1 Submarine Dive

Cases 1 and 2 produce results that are very similar and require approximately the same computation time. Since Case 1 is more flexible and easier to use, it should be used whenever the stern plane angle is known. In those maneuvers given in terms of ${}_0\dot{D}$ monitore by time or ${}_0D$, Case 3 gives results almost as good as Case 2, is easier to use, and takes less time. Case 4 should be used whenever ${}_0D$ is constant for any sizable duration and when the abrupt change in dive angle will not have any undesirable results. The advantage of Case 4 is its very small running time per iteration.

Thus, Case 1 is used for operator control, Case 3 for command and instructor control, and Case 4 for program control or those instructor control situations where the submarine can move as a point rather than a rigid body and when time is very important.

6.2.3.2 Aircraft Climb

The aircraft climb model can be used in a simple model for an aircraft as a target. Although not very realistic during transitional phases, it gives a good representation of the rigid-body orientation of the aircraft when v_c is fixed.

SECTION VII

RECOMMENDATIONS

Recommendations are made concerning both the use and the improvement of the kinematic vehicle simulation equations developed in this report.

These equations comprise a third category of vehicle simulation equations, falling between the two that already exist with respect to both fidelity and computer requirements. There are various types of simulation situations where it will be advantageous to use models in this category. These situations, described in section 6.2, use the concepts of operator control, command control, instructor control, and program control described in section 3.1.2.

Recommendations are made for two kinds of improvements of the models developed in this report. In the first place, several mathematical derivations need re-examination. Secondly, the values of vehicle constants in the submarine and surface vessel equations must be re-evaluated.

Several of the kinematic equations give results which are inconsistent with the high accuracy of the others. Time delays are in error for two of the submarine models. The time delay for speed in a deceleration maneuver is too small; the time delay for turn rate in a turn maneuver is too large. An error in time delay affects Cases 3 and 4. The time delay for speed loss in a surface vessel turn is somewhat smaller than it should be. Finally, improvement can be made in the kinematic models for aircraft acceleration and for aircraft climb. Recommendations for improving all of these models are discussed in detail in section 5.3. In section 3.5, it is shown that several values of the submarine constants in Reference 1 are erroneous. Furthermore, statements are made in Reference 1 to the effect that the values of the constants are not functions of the hydrodynamic partial derivatives used in the more complete dynamic equations. Instead, they are said to depend on the structure of the mathematical model as well as on the vehicle being simulated. Specific reference was made to the fact that, if a different integration technique were used to update the dynamic equations or if the dynamic equations were solved in a different order, then the values of these constants would very likely be different.

In the derivation of kinematic equations from dynamic equations, many changes are made from the original numerical approach. This should cause significant changes in the values of the constants. The constants, therefore, should be re-evaluated directly from the tactical data.

It is recommended, therefore, that methods be developed for direct evaluation of the constants of the vehicle's motion. This will be useful for improving the models developed for this report, and will provide a uniform and reliable method of constructing kinematic models for any submarine or surface vessel for which tactical data is available now or in the future.

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APPENDIX A

DERIVATIONS

A.1 SUBMARINE

The following equations describe the motion of a submarine. The symbols $A_1, \cdots, A_{11}$ represent certain constants which vary in value from one submarine to the next, depending on the handling characteristics of each.

$$\dot{S} = A_1\left\{{}_oS - (1 + A_2|\delta_r|)S\right\}\left\{{}_oS + (1 + A_2|\delta_r| + A_3)S\right\} \tag{1}$$

$$\ddot{C} = -(A_4 S\dot{C} + A_5\dot{C}|\dot{C}| + A_6 S^2\delta_r) \tag{2}$$

$$\ddot{D} = -(A_7 S\dot{D} + A_8\dot{D}|\dot{D}| + A_9 D + A_{11} S^2\delta_s) \tag{3}$$

The units chosen for the constants $(A_1, \cdots, A_{11})$ provide for speed in yards per second, angles in degrees, and angle rates in degrees per second.

S = Speed

${}_oS$ = Ordered speed, (forward speed to which the engine power output corresponds)

δ_r = Rudder angle (positive when rudder moves to the left)

C = Course angle (projection of the direction of the submarine's motion on the horizontal plane. The positive direction is clockwise, from north to east. $C = 0$ when the submarine is headed due north)

D = Dive angle (elevation angle from the horizontal. When the submarine is diving, D is negative)

δ_s = Stern plane angle (positive when the trailing edge moves down)

The rectangular coordinate system is left-handed, the X-axis points east, the Y-axis, north, and the Z-axis, down. $\dot{X}$, $\dot{Y}$ and $\dot{Z}$ are given by

$$\dot{X} = S \cos D \sin C$$

$$\dot{Y} = S \cos D \cos C$$

$$\dot{Z} = -S \sin D$$

A.1.1 Submarine Turn

The submarine turn equation will be solved first since portions of this solution will be used to solve submarine speed equations.

A.1.1.1 Solution of the Differential Equation

Equation 2 is the differential equation for $\dot{C}$, which is the rate of the submarine's course change

$$\ddot{C} = -(A_4 S\dot{C} + A_5 \dot{C}|\dot{C}| + A_6 S^2 \delta_r)$$

When $\dot{C}$ is positive, the equation is written

$$\ddot{C} = -(A_4 S\dot{C} + A_5 \dot{C}^2 + A_6 S^2 \delta_r)$$

When $\dot{C}$ is negative, it becomes

$$\ddot{C} = -(A_4 S\dot{C} - A_5 \dot{C}^2 + A_6 S^2 \delta_r)$$

NOTE: A_4, A_5 and A_6 are always positive.

The graph of $\ddot{C}$ versus $\dot{C}$ (Figure A-1) is comprised of two parabolic sections joined at point 0, $-A_6 S^2 \delta_r$. The curve has no maximum or minimum since the slope, $-(A_4 S + 2A_5|\dot{C}|)$, will always be negative.

In order to proceed with the solution of the differential equation, $\ddot{C}$ must be expressed as an analytic function of $\dot{C}$. It is doubtful that sufficient accuracy would be gained by the use of a cubic rather than a straight line to approximate the $\dot{C}|\dot{C}|$ term to warrant the added difficulty in integrating such a function. Therefore, a straight line will be used.

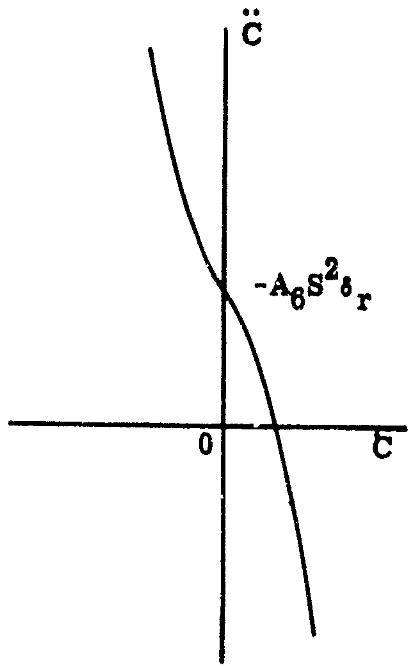

Figure A-1

The straight line will go through 0, $-A_6 S^2 \delta_r$ and the point on the curve corresponding to the steady-state turn rate.

In the types of maneuvers to which these formulas apply, there will be only one principal value of turn rate (${}_0\dot{C}$, the ordered turn rate) for each maneuver. This value c $\dot{C}$ will be the steady-state value, that is, $\ddot{C} = 0$ for $\dot{C} = {}_0\dot{C}$. The corresponding rudder a will be denoted by $\underline{\delta_r}$. The relationship between ${}_0\dot{C}$ and $\underline{\delta_r}$ can be found by letting $\ddot{C} = 0$.

$$
{}_0\dot{C} = \left(\frac{-A_4 \pm \sqrt{A_4^2 - 4A_5A_6\underline{\delta_r}}}{2A_5} \right) S
$$

for positive ${}_0\dot{C}$ and

$$
{}_0\dot{C} = \left(\frac{-A_4 \pm \sqrt{A_4^2 + 4A_5A_6\underline{\delta_r}}}{-2A_5} \right) S
$$

for negative ${}_0\dot{C}$

A glance at the graph (Figure A-1) will show that, for ${}_0\dot{C}$ positive, $\underline{\delta_r}$ is negative and for ${}_0\dot{C}$ negative, $\underline{\delta_r}$ is positive. This is because the graph has a negative slope. Al ${}_0\dot{C}$ and $-A_6S^2\underline{\delta_r}$ are $\dot{C}$ and $\ddot{C}$ intercepts respectively. Therefore both radicals can be written

$$
\sqrt{A_4^2 + 4A_5A_6|\underline{\delta_r}|}
$$

The radical is greater than A_4 (since A_4, A_5 and A_6 are all positive). In order to have the proper sign for ${}_0\dot{C}$ in equations 4 and 5 , the positive sign will be used for radicals in both cases. The two formulas can then be combined as follows:

$$
{}_0\dot{C} = \frac{-\underline{\delta_r}S}{|\underline{\delta_r}|} \; \frac{\sqrt{A_4^2 + 4A_5A_6|\underline{\delta_r}|} - A_4}{2A_5}
$$

The final expression for ${}_0\dot{C}$ can be found by substituting the steady-state value of S in th turn. The steady-state value of S can be determined by setting $\dot{S}$ to 0 in equation 1 . S $1 + A_2|\delta_r| + A_3$ will always be positive, ${}_0S - (1 + A_2|\delta_r|)S = 0$. Hence steady-state S, de noted by S_f, is given by

$$
S_f = \frac{{}_0S}{1 + A_2|\delta_r|}
$$

and

$$
{}_0\dot{C} = -\frac{\underline{\delta_r}}{|\underline{\delta_r}|} \; {}_0S \, \frac{\sqrt{A_4^2 + 4A_5A_6|\underline{\delta_r}|} - A_4}{2A_5(1 + A_2|\underline{\delta_r}|)}
$$

The straight line approximating the graph of equation 2 can now be calculated. The slope of the line will be given in terms of $\underline{\delta_r}$, the principal value of δ_r in a maneuver, and S, the present value of speed. The $\ddot{C}$ intercept will be $-A_6S^2\underline{\delta_r}$.

Since the slope is determined by the current value of S rather than S_f, equation 6 will be used for the intercept rather than equation 8 .

When $\delta_r = \underline{\delta_r}$, the line must go through the two points 0, $-A_6S^2\underline{\delta_r}$ and

$$\frac{-\underline{\delta_r}S}{|\underline{\delta_r}|}\;\frac{\sqrt{A_4^2 + 4A_5A_6|\underline{\delta_r}|} - A_4}{2A_5}, 0.$$

Denote $-\dfrac{\underline{\delta_r}}{|\underline{\delta_r}|}\,\dfrac{\sqrt{A_4^2 + 4A_5A_6|\underline{\delta_r}|} - A_4}{2A_5}$ by F_c (9)

Therefore, the slope of the line is $\dfrac{A_6S^2\underline{\delta_r}}{SF_c}$. The slope of the line remains constant. However, the line can be displaced vertically if $-A_6S^2\delta_r$, the $\ddot{C}$ intercept, changes in value. Thus, using the y = mx + b form of the equation for a straight line, the approximating formula is

$$\ddot{C} = \frac{A_6S^2\underline{\delta_r}\dot{C}}{SF_c} - A_6S^2\delta_r \tag{10}$$

This is a linear differential equation for $\dot{C}$ and is easily solved. The solution will be erroneous however, unless δ_r is represented as a function of time. δ_r is most easily approximated by a linear function of time. Let $\delta_r = \delta_1 + \dot{\delta}_r t$, where $\delta_1 = \delta_r$ at time t = 0.

The solution of the homogeneous equation is $\dot{C} = k \exp\left[A_6S\underline{\delta_r}t/F_c\right]$. Let $\dot{C} = y_1 + y_2$ be a particular solution. Thus,

$$0 = y_2 - A_6S\left[(\underline{\delta_r}(y_1 + y_2t)/F_c) - S(\delta_1 + \dot{\delta}_r t)\right]$$

This must be true for all t, therefore, letting t = 0,

$$0 = y_2 - A_6S\left[(\underline{\delta_r}y_1/F_c) - S\delta_1\right]$$

Dropping this sum from the equation and letting t = 1 produces

$$0 = A_6S\left[(\underline{\delta_r}y_2/F_c) - S\dot{\delta}_r\right]$$

Hence,

$$y_2 = SF_c\dot{\delta}_r/\underline{\delta_r}$$

$$y_1 = \frac{F_c}{\underline{\delta_r}}\left[\frac{\dot{\delta}_r F_c}{\underline{\delta_r} A_6} + \delta_1 S\right]$$

Therefore the general solution is

$$\dot{C} = k\,\exp[A_6 S\underline{\delta_r}t/F_c] + \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c\dot{\delta}_r}{\underline{\delta_r}A_6} + \delta_1 S + S\dot{\delta}_r t\right]$$

However, since $\delta_r = \delta_1 + \dot{\delta}_r t$,

$$\dot{C} = k\,\exp[A_6 S\underline{\delta_r}t/F_c] + \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c\dot{\delta}_r}{\underline{\delta_r}A_6} + S\delta_r\right]$$

Let $\dot{C}$ at time $t = nh$ be written $\dot{C}_n$ for all n.

Then

$$\dot{C}_n - \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c\dot{\delta}_r}{A_6\underline{\delta_r}} + S\delta_{r_n}\right] = k\,\exp[A_6 S\underline{\delta_r}nh/F_c]$$

$$\dot{C}_{n-1} - \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c\dot{\delta}_r}{A_6\underline{\delta_r}} + S\delta_{r_{n-1}}\right] = k\,\exp[A_6 S\underline{\delta_r}nh/F_c]\exp[-A_6 S\underline{\delta_r}h/F_c]$$

Therefore

$$\dot{C}_n - \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c\dot{\delta}_r}{A_6\underline{\delta_r}} + S\delta_{r_n}\right] = \left(\dot{C}_{n-1} - \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c\dot{\delta}_r}{A_6\underline{\delta_r}} + S\delta_{r_{n-1}}\right]\right)\exp\left[\frac{A_6 S\underline{\delta_r}h}{F_c}\right] \quad (11$$

A.1.1.2 Operator and Command Control Situations

The instructor and program control situations will be developed after the speed equations.

Operator Control

In the operator control situation, δ_r and $\underline{\delta_r}$ are direct inputs. Therefore, the only approximation required is

$$\exp[A_6\underline{\delta_r}Sh/F_c] \approx 1 + A_6\underline{\delta_r}Sh/F_c \quad (1$$

The validity of this approximation depends on the size of $A_6 \underline{\delta_r} Sh/F_c$. $\underline{\delta_r}/F_c$ increases roughly as $\sqrt{|\underline{\delta_r}|}$, as shown by equation 9. The largest value of $\underline{\delta_r}$ that is likely to be encountered is 35°. According to available documentation, A_4, A_5 and A_6 will vary considerably. Their values are given for three different vessels. Using these figures, the three different maximum values for Sh such that $(A_6 \underline{\delta_r} Sh/F_c) < 1$ when $|\underline{\delta_r}| = 35°$ are given by

Submarine I Sh < 64.4 yards = 114 knot-seconds

Submarine II Sh < 2,373 yards = 4,215 knot-seconds

Submarine III Sh < 139 yards = 246 knot-seconds

When h equals 1.0 seconds, $A_6 \underline{\delta_r} Sh/F_c < 1$ for any possible submarine speed. Proceeding with the approximation, equations 11 and 12 provide

$$\dot{C}_n - \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c \dot{\delta}_r}{A_6 \underline{\delta_r}} + S\delta_{r_n}\right] = \left(\dot{C}_{n-1} - \frac{F_c}{\underline{\delta_r}}\left[\frac{F_c \dot{\delta}_r}{A_6 \underline{\delta_r}} + S\delta_{r_{n-1}}\right]\right)\left(1 + \frac{A_6 S \underline{\delta_r} h}{F_c}\right)$$

$$\dot{C}_n - \dot{C}_{n-1} = \frac{F_c}{\underline{\delta_r}} S(\delta_{r_n} - \delta_{r_{n-1}}) - \frac{F_c \dot{\delta}_r Sh}{\underline{\delta_r}} + \frac{A_6 S \underline{\delta_r} h}{F_c}\left(\dot{C}_{n-1} - \frac{F_c S \delta_{r_{n-1}}}{\underline{\delta_r}}\right)$$

but $\dot{\delta}_r h = \delta_{r_n} - \delta_{r_{n-1}}$

Therefore

$$\dot{C}_n - \dot{C}_{n-1} = \frac{A_6 S \underline{\delta_r} h}{F_c}\left(\dot{C}_{n-1} - \frac{F_c S \delta_{r_{n-1}}}{\underline{\delta_r}}\right)$$

The final assumption used in the above is that the variation of S over the interval h will be small enough to enable the equation to hold with the same value of S in all terms.

$$\dot{C}_n - \dot{C}_{n-1} = A_6 S_{n-1} h[(\underline{\delta_r} \dot{C}_{n-1}/F_c) - S_{n-1}\delta_{r_{n-1}}] \tag{13}$$

S_{n-1} is used rather than S_n because it is larger and will therefore lead to a larger value of $\dot{C}_n - \dot{C}_{n-1}$. This compensates somewhat for the fact that the straight line produces slightly smaller values for $|\ddot{C}|$ than would be produced by the parabola.

Command Control

In the command control situation, ${}_oS$ and ${}_o\dot{C}$ (equation 8) are the only inputs to the equations other than T. T equals the time it takes the rudder to move to $\underline{\delta_r}$.

There will be a command control equation for S, so that S_n will be known at each iteration. S_f, defined by equation 7, can be found as a function of ${}_o\dot{C}$ using equation 6. Equation 7 will be used to determine $\underline{\delta_r}$, which is then inserted in equation 8 with $S = S_f$. Equation 8 is then solved for S_f.

$$\left|\underline{\delta_r}\right| = \left(\frac{{}_oS}{S_f} - 1\right)\frac{1}{A_2} \tag{14}$$

$$\left|{}_o\dot{C}\right| = \frac{S_f}{2A_5}\left(\sqrt{A_4^2 + 4A_5A_6\left(\frac{{}_oS}{S_f} - 1\right)\frac{1}{A_2}} - A_4\right)$$

$$\left(2A_5\left|{}_o\dot{C}\right| + A_4S_f\right)^2 = S_f^2\left(A_4^2 + 4A_5A_6\left[\frac{{}_oS}{S_f} - 1\right]\frac{1}{A_2}\right)$$

$${}_o\dot{C}^2A_5 + A_4S_f\left|{}_o\dot{C}\right| = (A_6/A_2)({}_oSS_f - S_f^2)$$

$$S_f^2 + S_f\left(\frac{A_2A_4\left|{}_o\dot{C}\right|}{A_6} - {}_oS\right) + \frac{A_2A_5}{A_6}\,{}_o\dot{C}^2 = 0$$

$$S_f = \left\{{}_oS - \frac{A_2A_4\left|{}_o\dot{C}\right|}{A_6} + \sqrt{\left({}_oS - \frac{A_2A_4\left|{}_o\dot{C}\right|}{A_6}\right)^2 - \frac{4A_2A_5{}_o\dot{C}^2}{A_6}}\right\}^{1/2} \tag{15}$$

where the radical is taken to be positive so that $S_f = {}_oS$ when ${}_o\dot{C} = 0$.

With S_f known (in terms of ${}_o\dot{C}$), F_c can be found.

$$F_c = {}_o\dot{C}/S_f \tag{16}$$

Equation 10 can now be written in terms of S_f and ${}_o\dot{C}$ where $\delta_r = \underline{\delta_r}$. $\underline{\delta_r}$ is given by equation 14. The sign of $\underline{\delta_r}$ will be the opposite of the sign of ${}_o\dot{C}$.

$$\ddot{C} = -\frac{A_6S^2\left(\frac{{}_oS}{S_f} - 1\right)\frac{\dot{C}}{A_2}}{S\left|{}_o\dot{C}\right|/S_f} + A_6S^2\left(\frac{{}_oS}{S_f} - 1\right)\frac{{}_o\dot{C}}{\left|{}_o\dot{C}\right|A_2}$$

$$\ddot{C} = \frac{A_6S}{A_2\left|{}_o\dot{C}\right|}\left(\frac{{}_oS}{S_f} - 1\right)\left({}_o\dot{C}S - S_f\dot{C}\right) \tag{17}$$

$$\frac{-1}{S_f}\log({}_o\dot{C}S - S_f\dot{C}) = \frac{A_6 St}{A_2|{}_o\dot{C}|}\left(\frac{{}_oS}{S_f} - 1\right) + K_3$$

$${}_o\dot{C}S - S_f\dot{C} = K_2 \exp\left[-\frac{A_6 St}{A_2|{}_o\dot{C}|}({}_oS - S_f)\right] \tag{18}$$

$${}_o\dot{C}S_n - S_f\dot{C}_n = K_2 \exp\left[-\frac{A_6 S_n nh}{A_2|{}_o\dot{C}|}({}_oS - S_f)\right]$$

for all n.

Assuming again that $S_n \approx S_{n-1}$,

$$({}_o\dot{C}S_{n-1} - S_f\dot{C}_n) = ({}_o\dot{C}S_{n-1} - S_f\dot{C}_{n-1})\exp\left[-\frac{A_6 S_{n-1}h}{A_2|{}_o\dot{C}|}({}_oS - S_f)\right]$$

Note that this is the same as equation 11 after making the substitution indicated in equations 14 and 16.. Therefore the expansion of the exponential holds for the same range of values of S as did the expansion of the exponential in equation 11 .

$$\dot{C}_n - \dot{C}_{n-1} = \frac{-hA_6 S_{n-1}({}_oS - S_f)}{A_2|{}_o\dot{C}|S_f}(S_f\dot{C}_{n-1} - {}_o\dot{C}S_{n-1}) \tag{19}$$

This can be rewritten in a simpler form

$$\frac{{}_oS - S_f}{A_2 S_f} = |\underline{\delta_r}| \text{ by equation 14}$$

Furthermore, $\underline{\delta_r}$ and ${}_o\dot{C}$ always have opposite signs, therefore

$$|\underline{\delta_r}|/|{}_o\dot{C}| = -\underline{\delta_r}/{}_o\dot{C}\ . \tag{20}$$

Also, when $S = S_f$ and $\dot{C} = {}_o\dot{C}$, $\ddot{C}$ in equation 2 will be zero.

$$A_4 S_{f\,o}\dot{C} + A_{5\,o}\dot{C}|{}_o\dot{C}| + A_6 S_f^2\,\underline{\delta_r} = 0$$

$$-\frac{A_6 S_f^2 \underline{\delta_r}}{{}_o\dot{C}} = A_4 S_f + A_5 \left|{}_o\dot{C}\right|$$

$$\frac{A_6 S_{n-1}({}_oS - S_f)}{A_2\left|{}_o\dot{C}\right| S_f} = \frac{A_6 S_{n-1}\left|\underline{\delta_r}\right|}{\left|{}_o\dot{C}\right|} = \frac{-A_6 S_{n-1}\underline{\delta_r}}{{}_o\dot{C}}$$

$$= (A_4 S_f + A_5\left|{}_o\dot{C}\right|)\frac{S_{n-1}}{S_f^2} \tag{21}$$

$$\dot{C}_n - \dot{C}_{n-1} = \frac{-hS_{n-1}}{S_f^2}(S_f\dot{C}_{n-1} - {}_o\dot{C}S_{n-1})(A_4 S_f + A_5\left|{}_o\dot{C}\right|) \tag{22}$$

This form eliminates the indeterminate 0/0 that occurs in equation 19 when $\underline{\delta_r} = 0$, causing both ${}_oS - S_f$ and $\left|{}_o\dot{C}\right|$ to be zero.

The abrupt introduction of $\underline{\delta_r}$ as the rudder angle rather than building it up from $\delta_r = 0$ will be compensated for by letting $\dot{C}_n = 0$ for $nh < T/2$ and using equation 22 when $nh \geq T/2$. T is the estimated time for the rudder to build up to $\underline{\delta_r}$ from zero and may be determined by using the following formula.

Let $\dot{\delta}_r$ be the estimated rudder deflection angle rate in degrees, per second.

Therefore, using equation 14

$$T = \left(\frac{{}_oS}{S_f} - 1\right)\frac{1}{A_2\left|\dot{\delta}_r\right|} \tag{23}$$

A.1.2 Submarine Speed

A.1.2.1 Solution of the Differential Equation

Equation 1 is used to determine S, the speed of a submarine. It may be solved directly.

$$\frac{dS}{A_1\left\{{}_oS - (1 + A_2|\delta_r|)S\right\}\left\{{}_oS + (1 + A_2|\delta_r| + A_3)S\right\}} = dt$$

Using partial fractions,

$$dS\left\{\frac{A}{{}_oS-(1+A_2|\delta_r|)S}+\frac{B}{{}_oS+(1+A_2|\delta_r|+A_3)S}\right\}=A_1\,dt$$

$$A=\frac{1+A_2|\delta_r|}{{}_oS[2(1+A_2|\delta_r|)+A_3]}$$

$$B=\frac{1+A_2|\delta_r|+A_3}{{}_oS[2(1+A_2|\delta_r|)+A_3]}$$

$$\frac{{}_oS+(1+A_2|\delta_r|+A_3)S}{{}_oS-(1+A_2|\delta_r|)S}=K_1\exp[{}_oSA_1t(2[1+A_2|\delta_r|]+A_3)] \qquad (24)$$

$$\frac{{}_oS+(1+A_2|\delta_r|+A_3)S_n}{{}_oS-(1+A_2|\delta_r|)S_n}=\frac{{}_oS+(1+A_2|\delta_r|+A_3)S_{n-1}}{{}_oS-(1+A_2|\delta_r|)S_{n-1}}$$

$$\times\exp\left[{}_oSA_1h(2[1+A_2|\delta_r|]+A_3)\right]$$

(Refer to equation 11).

This can be solved for $S_n - S_{n-1}$ in terms of S_{n-1}.

$$S_n-S_{n-1}=\frac{\left[{}_oS-(1+A_2|\delta_r|)S_{n-1}\right]\left[{}_oS+(1+A_2|\delta_r|+A_3)S_{n-1}\right]\left[\exp({}_oSA_1h[2(1+A_2|\delta_r|)+A_3])-1\right]}{\left\{\begin{array}{l}(1+A_2|\delta_r|+A_3)[{}_oS-(1+A_2|\delta_r|)S_{n-1}]\\ +(1+A_2|\delta_r|)[{}_oS+(1+A_2|\delta_r|+A_3)S_{n-1}]\exp\left({}_oSA_1h[2(1+A_2|\delta_r|)+A_3]\right)\end{array}\right\}}$$

This expression is still much too cumbersome for practical use, therefore certain approximations will be made. First, the exponential will be replaced by the first two terms of its Taylor expansion.

$$\exp\left({}_oSA_1h[2(1+A_2|\delta_r|)+A_3]\right)\approx 1+{}_oSA_1h[2(1+A_2|\delta_r|)+A_3]$$

The first two terms will be a good approximation for the exponential, provided that the argument of the exponential is small. The argument is largest for $|\delta_r| = 35°$, or full rudder. Using this value, the maximum values of ${}_oSh$ such that the argument will be less than one, corresponding to the three submarines for which figures are available, are as follows.

NOTE: The values given for A_2 are 0.275, 0.203 and 0.0403. These are incorrect. However, since A_2 is easy to compute, the following values have been used instead: 0.0275, 0.0203 and 0.0271. The values of A_1 and A_3 appear to be correct. (Refer to paragraph 3.5)

Submarine I $\quad {}_0Sh < 98.4$ yards = 174 knot-seconds

Submarine II $\quad {}_0Sh < 117$ yards = 208 knot-seconds

Submarine III $\quad {}_0Sh < 118$ yards = 210 knot-seconds

When ${}_0Sh$ satisfies the appropriate condition, the first two terms of the expansion are used.

$$S_n - S_{n-1} = \frac{[{}_0S - (1 + A_2|\delta_r|)S_{n-1}]\,[{}_0S + (1 + A_2|\delta_r| + A_3)S_{n-1}]\,A_1h}{1 + A_1h(1 + A_2|\delta_r|)[{}_0S + (1 + A_2|\delta_r| + A_3)S_{n-1}]}$$

The next step is to put the denominator of the resulting fraction into the form $1 + x$, where $x < 1$. Then, the approximation $\frac{1}{1+x} \approx 1 - x$ will be used. The denominator can be expanded provided that

$$A_1h(1 + A_2|\delta_r|)[{}_0S + (1 + A_2|\delta_r| + A_3)S_{n-1}] < 1 \qquad (25)$$

This is slightly different from the inequality that must be satisfied in order to expand the exponential. In a turn, this expression will be greatest when $\delta_r = 35°$ and $S_{n-1} = {}_0S$. Using the same values of A_2, the maximum values become

Submarine I $\quad {}_0Sh < 72.0$ yards = 128 knot-seconds

Submarine II $\quad {}_0Sh < 93.2$ yards = 166 knot-seconds

Submarine III $\quad {}_0Sh < 86.4$ yards = 153 knot-seconds

In a deceleration maneuver, the above inequalities must be satisfied for original speed times h, rather than ${}_0Sh$.

When ${}_0Sh$ satisfies the condition appropriate to the particular submarine to be simulated, the approximation $\frac{1}{1+x} \approx 1 - x$ can be used. When used however, the h^2 term, when the numerator is multiplied by $1 + x$, has coefficient

$$[{}_0S - (1 + A_2|\delta_r|)S_{n-1}][{}_0S + (1 + A_2|\delta_r| + A_3)S_{n-1}]^2A_1^2(1 + A_2|\delta_r|).$$

This will be much less than the coefficient of the h term unless the restrictive inequality 25 is very nearly an equality. Therefore this term will be omitted.

The final formula is therefore

$$S_n - S_{n-1} = A_1 h[{}_oS - (1 + A_2|\delta_r|)S_{n-1}][{}_oS + (1 + A_2|\delta_r| + A_3)S_{n-1}] \tag{26}$$

A.1.2.2 Four Control Situations

Operator Control

Equation 26 can be used, without modification, for the operator controlled turn. In an operator controlled acceleration or deceleration maneuver, $\delta_r = 0$ and equation 21 becomes

$$S_n - S_{n-1} = A_1 h({}_oS - S_{n-1})({}_oS + [1 + A_3]S_{n-1}) \tag{27}$$

Command Control

Equation 27 is suitable for the command controlled acceleration or deceleration maneuver. In the command controlled turn maneuver however, δ_r will not be one of the inputs. Instead, the input will be S_f, which is derived from ${}_o\dot{C}$ by equation 15 . The relationship between δ_r and S_f is provided by equation 7 . It is this equation which will be used to express $S_n - S_{n-1}$ in terms of S_f.

$$S_f = {}_oS/(1 + A_2|\underline{\delta_r}|)$$

$$1 + A_2|\underline{\delta_r}| = {}_oS/S_f$$

Substituting in equation 26 ,

$$S_n - S_{n-1} = {}_oSA_1 h\left[1 - \frac{S_{n-1}}{S_f}\right]\left[{}_oS + \left(\frac{{}_oS}{S_f} + A_3\right)S_{n-1}\right]$$

$$S_n - S_{n-1} = \frac{{}_oS}{S_f} A_1 h(S_f - S_{n-1})\left[{}_oS + \left(\frac{{}_oS}{S_f} + A_3\right)S_{n-1}\right] \tag{28}$$

Note that equation 28 is the same as equation 27 when S_f is equal to ${}_oS$.

In the command control situation, the buildup of δ_r from zero to $\underline{\delta_r}$ is accounted for by letting $S_n = {}_oS$ for $nh < T/2$ and then using equation (28) when $nh \geq T/2$. T equals the time required for the rudder to move from zero to $\underline{\delta_r}$. (Refer to equation 23.)

Instructor Control

In the instructor controlled situation, $S_n - S_{n-1}$ is expressed as a function of $_oS$ and S_f, but not S_{n-1}. This is accomplished by equating the following integrals and taking the limit as $x \to \infty$.

$$\int_0^{(S_f-S_o)/m} (S_o + mt)dt + \int_{(S_f-S_o)/m}^{x} S_f\, dt = \int_0^{x} S\, dt \qquad (29)$$

where $S_o + mt = S$ so $S_n - S_{n-1} = mh$.

Before equation 29 can be solved for m, an expression for S must be found that can be readily integrated. Such an expression for S is implicit in equation 24 . There are no approximations involved in its integration. K_1 will be evaluated by letting $S = S_o$ when $t = 0$ In this way the resulting expression can be used for either turn or acceleration/deceleratior maneuvers.

$$K_1 = \frac{_oS + (1 + A_2|\delta_r| + A_3)S_o}{_oS - (1 + A_2|\delta_r|)S_o} \qquad (30)$$

The following notations are provided to facilitate the performance of subsequent calculations.

$$a = (1 + A_2|\delta_r| + A_3)$$
$$b = (1 + A_2|\delta_r|)$$
$$c = {_oS}A_1(a + b) \qquad (31)$$

Therefore, equation 24 can now be written

$$\frac{_oS + aS}{_oS - bS} = K_1 e^{ct}$$

$$S = \frac{_oS(K_1e^{ct} - 1)}{a + bK_1e^{ct}} = \frac{_oS}{b}\left[\frac{bK_1e^{ct} + a - (a+b)}{bK_1e^{ct} + a}\right]$$

$$S = \frac{_oS}{b}\left[1 - \frac{a+b}{bK_1e^{ct} + a}\right]$$

$$\int_0^x S\, dt = \frac{_oS}{b}x - \frac{(a+b)_oS}{b}\int_0^x \frac{dt}{bK_1e^{ct} + a}$$

$$\int_0^x \frac{dt}{bK_1e^{ct} + a} = \frac{1}{ac}\,[ct - \log(bK_1e^{ct} + a)]_0^x$$

$$\int_0^x S\,dt = {}_oS[\frac{1}{b} - \frac{(a+b)}{ab}]x + {}_oS\frac{(a+b)}{abc}\log\left[\frac{bK_1e^{cx} + a}{bK_1 + a}\right]$$

$$= {}_oS\left[\frac{-x}{a} + \frac{a+b}{abc}(\log[bK_1e^{cx} + a] - \log[bK_1 + a])\right]$$

As $x \to \infty$,

$$\log(bK_1e^{cx} + a) \to cx + \log bK_1$$

So,

$$\int_0^x S\,dt \to {}_oS\left[\frac{-x}{a} + \frac{a+b}{abc}(cx - \log[1 + \frac{a}{bK_1}])\right] = {}_oS\left[\frac{x}{b} - \frac{a+b}{abc}\log(1 + \frac{a}{bK_1})\right] \qquad (32)$$

Using equation 29 , this equals

$$S_o\left[\frac{S_f - S_o}{m}\right] + \frac{m(S_f - S_o)^2}{2m^2} + S_f\left[x - \frac{(S_f - S_o)}{m}\right]$$

$$= -\frac{(S_f - S_o)^2}{m} + \frac{(S_f - S_o)^2}{2m} + S_fx = S_fx - \frac{(S_f - S_o)^2}{2m}$$

Thus, equation 29 becomes

$$S_fx - \frac{(S_f - S_o)^2}{2m} = \frac{{}_oS}{b}x - \frac{{}_oS(a+b)}{abc}\log(1 + \frac{a}{bK_1})$$

In an acceleration/deceleration maneuver, $b = 1$ and $S_f = {}_oS$. In a turn, $S_f = {}_oS/b$ by equations 31 and 14 . Therefore, the terms containing x drop out in both cases.

Using equation 31 ,

$${}_oS\,\frac{a+b}{abc} = \frac{1}{A_1ab}$$

Thus,

$$m = +\frac{1}{2}(S_f - S_o)^2\left[\frac{1}{A_1ab}\log\left(1 + \frac{a}{bK_1}\right)\right]^{-1} \qquad (33)$$

Equation 33 still has to be interpreted separately for turn and acceleration/deceleration. In a turn, $S_o = {}_oS$, therefore $K_1 = (1 + a)/(1 - b)$ and the argument of the logarithm becomes

$$1 + \frac{a(1 - b)}{b(1 + a)}$$

a and b are both positive, therefore $a < 1 + a$ and $|1 - b| < b$ and $|a(1 - b)| < b(1 + a)$. Thus, the logarithm can be expanded

$$\log\left(1 + \frac{a(1 - b)}{b(1 + a)}\right) \approx \frac{a(1 - b)}{b(1 + a)}$$

$$m_t = (S_f - {}_oS)^2 \frac{b^2 A_1 (1 + a)}{2(1 - b)}$$

a and b are then expressed in terms of ${}_oS$ and S_f, using equations 31 and 14 .

$$a = \frac{{}_oS}{S_f} + A_3$$

$$b = \frac{{}_oS}{S_f}$$

$$1 - b = (S_f - {}_oS)/S_f$$

$$m_t = (S_f - {}_oS)\left(\frac{{}_oS^2}{S_f}\right)\left(\frac{A_1}{2}\right)\left(\frac{{}_oS}{S_f} + A_3 + 1\right) \qquad (34)$$

We next consider the acceleration deceleration maneuver.

$$a = 1 + A_3$$

$$b = 1$$

$$K_1 = \frac{{}_oS + (1 + A_3)S_o}{{}_oS - S_o} \text{ (equation 30)}$$

In this case $S_f = {}_oS$, so equation 33 can be rewritten as follows:

$$m_a = \frac{1}{2}({}_oS - S_o)^2\left[\frac{1}{A_1(1 + A_3)} \log\left(1 + \frac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o}\right)\right]^{-1}$$

A_3 is negative but greater than -1, therefore

$$\left|\frac{(1 + A_3)({}_oS - S_o)}{{}_oS + (1 + A_3)S_o}\right| < 1$$

provided that ${}_0S$ and S_0 are both positive. Expanding the logarithm

$$m_a = \frac{1}{2}({}_0S - S_0)^2 \frac{[{}_0S + (1 + A_3)S_0]A_1}{{}_0S - S_0}$$

$$m_a = ({}_0S - S_0)\frac{A_1}{2}[{}_0S + (1 + A_3)S_0] \tag{35}$$

The two formulations may be summarized as follows:

a. Turn

$$S_n = {}_0S \qquad \text{when } nh < T/2$$

$$S_n - S_{n-1} = \frac{hA_1}{2}(S_f - {}_0S)\frac{{}_0S^2}{S_f}\left(\frac{{}_0S}{S_f} + A_3 + 1\right)$$

$$\text{when } nh \geq T/2 \text{ and } S_n > S_f$$

$$S_n = S_f \text{ otherwise} \tag{36}$$

b. Acceleration

$$S_n - S_{n-1} = \frac{hA_1}{2}({}_0S - S_0)[{}_0S + (1 + A_3)S_0]$$

$$\text{when } (S_n - {}_0S)(S_0 - {}_0S) > 0$$

$$S_n = {}_0S \text{ otherwise} \tag{37}$$

Program Control

In the program control situation, speed changes occur abruptly. Realism is achieved by introducing a time delay so that the distance traversed at some future time (x) will be the same as if an exponential buildup were used.

The integrals to be equated are

$$\int_0^{\tau} S_0\, dt + \int_{\tau}^{x} S_f\, dt = \int_0^{x} S\, dt \tag{38}$$

as $x \to \infty$.

Equation 32 already has

$$\lim_{x \to \infty} \int_0^{x} S\, dt = {}_0S\left[\frac{x}{b} - \frac{a+b}{abc}\log\left(1 + \frac{a}{bK_1}\right)\right]$$

This is now equated to

$$(S_o - S_f)\tau + S_f x.$$

As mentioned following equation 32 , $\frac{{}_oS}{b} = S_f$ in both turn and acceleration/deceleration maneuvers, therefore the terms containing x drop out. Also $|a/bK_1| < 1$ (refer to discussions following equations 33 and 34) . Hence, the following approximation will be used for τ:

$$\tau = \frac{-{}_oS}{S_o - S_f}\ \frac{(a+b)a}{ab^2cK_1}$$

$$\tau = \frac{-1}{(S_o - S_f)b^2 A_1 K_1} \qquad (3$$

by using equation 31 .

In a turn, $K_1 = \frac{1+a}{1-b}$ refer to equations 30 and 31. However by using equations and 31 , this can be written

$$K_1 = \left(\frac{{}_oS}{S_f} + A_3 + 1\right) S_f/(S_f - {}_oS)$$

while

$$b = {}_oS/S_f$$

$$S_o = {}_oS$$

Hence,

$$\tau_t = S_f \Big/ {}_oS^2 A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right) \qquad (4$$

In an acceleration/deceleration maneuver,

$$b = 1$$

$$K_1 = \frac{{}_oS + (1 + A_3)S_o}{{}_oS - S_o}$$

$$S_f = {}_oS$$

therefore,

$$\tau_a = 1 \Big/ \left({}_oS + (1 + A_3)S_o\right) A_1 \qquad (4$$

A

In summation, the two formulations are as follows:

a. Turn

$$S_n = {}_oS \text{ when } nh < \frac{T}{2} + S_f \Big/ {}_oS^2 A_1 \left(\frac{{}_oS}{S_f} + A_3 + 1 \right)$$

$$S_n = S_f \text{ when } nh \geq \frac{T}{2} + S_f \Big/ {}_oS^2 A_1 \left(\frac{{}_oS}{S_f} + A_3 + 1 \right) \tag{42}$$

b. Acceleration

$$S_n = {}_oS \text{ when } nh < 1 \Big/ ({}_oS + [1 + A_3] S_o) A_1$$

$$S_n = {}_oS \text{ when } nh \geq 1 \Big/ ({}_oS + [1 + A_3] S_o) A_1 \tag{43}$$

A. 1. 3 Two More Control Situations for Submarine Turn

A. 1. 3. 1 The Integral of $\dot{C}$

To proceed with the instruction control and program control approximations of the submarine turn, it will be necessary to evaluate

$\lim_{x \to \infty} \int_o^x \dot{C}\, dt$. $\dot{C}$ will be expressed using equation 18, rewritten here as equation 44.

$${}_o\dot{C}S - S_f\dot{C} = K_2 \exp\left[\frac{-A_6 St}{A_2 |{}_o\dot{C}|} ({}_oS - S_f) \right] \tag{44}$$

K_2 is evaluated by letting $\dot{C} = 0$ and $S = {}_oS$ when $t = 0$.

$$K_2 = {}_o\dot{C}\,{}_oS$$

$$\dot{C} = \frac{{}_o\dot{C}}{S_f} \left[S - {}_oS \exp\left(\frac{-A_6 St}{A_2 |{}_o\dot{C}|} \; ({}_oS - S_f) \right) \right] \tag{45}$$

Equation 45 will be integrated from zero to infinity using a variable S. Equation 32 will be used to find the integral of $\frac{{}_o\dot{C}S}{S_f}$. The S in the argument of the exponential must be approximated by a simpler formula so that the integral can be evaluated. Equation 42 will be used for this.

$$\text{Let } \tau_1 = S_f / {}_oS^2 A_1 \left(\frac{{}_oS}{S_f} + A_3 + 1 \right)$$

Therefore,

$$\lim_{x\to\infty}\int_0^x \dot{C}\,dt = \frac{{}_o\dot{C}}{S_f}\left[\lim_{x\to\infty}\int_0^x S\,dt - {}_oS\int_0^{\tau_1}\exp\left[\frac{-A_{6o}St({}_oS - S_f)}{A_2|{}_o\dot{C}|}\right]dt\right.$$

$$\left. -\lim_{x\to\infty} {}_oS\int_{\tau_1}^{x}\exp\left[\frac{-A_6 S_f t({}_oS - S_f)}{A_2|{}_o\dot{C}|}\right]dt\right] \tag{46}$$

$$\lim_{x\to\infty}\int_0^x S\,dt = S_f x - S_f(S_f - {}_oS)\Big/{}_oS^2 A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)$$

from equations 29, 32, 33, and 34

$$\int_0^{\tau_1}\exp\left[\frac{-A_{6o}St({}_oS - S_f)}{A_2|{}_o\dot{C}|}\right]dt + \lim_{x\to\infty}\int_{\tau_1}^{x}\exp\left[\frac{-A_6 S_f t({}_oS - S_f)}{A_2|{}_o\dot{C}|}\right]dt$$

$$= \frac{-A_2|{}_o\dot{C}|}{A_6({}_oS - S_f)}\left\{\frac{1}{{}_oS}\left[\exp\left(\frac{-A_{6o}S\tau_1({}_oS - S_f)}{A_2|{}_o\dot{C}|}\right) - 1\right]\frac{-1}{S_f}\exp\left(\frac{-A_6 S_f\tau_1({}_oS - S_f)}{A_2|{}_o\dot{C}|}\right)\right\}$$

$\frac{{}_oS - S_f}{A_2} = S_f|\delta_r|$ by using equation 14 , and $\frac{S_f}{|{}_o\dot{C}|} = \frac{1}{|F_c|}$ by using equation 16 . Hence, the arguments of the two exponentials become $\frac{-A_{6o}S\tau_1|\delta_r|}{|F_c|}$ and $\frac{-A_6 S_f\tau_1|\delta_r|}{|F_c|}$ respectively.

The maximum values of ${}_oS\tau_1$ and $S_f\tau_1$ for these quantities, to have magnitude less than one, are provided in the discussion following equation 12 for $|\delta_r| = 35°$. τ is approximately 10 to 20 seconds, therefore, unless S is quite small, the arguments will have magnitude greater than one. However, the difference of the two exponentials will be used and th pansion will be carried out to three terms; hence, the resultant error will be smal

$$\frac{1}{{}_oS}\exp\left(\frac{A_{6o}S\tau_1\delta_r}{F_c}\right) - \frac{1}{S_f}\exp\left(\frac{A_6 S_f\tau_1\delta_r}{F_c}\right)$$

$$= \left(\frac{1}{{}_oS} - \frac{1}{S_f}\right) + \frac{A_6^2\tau_1^2\delta_r^2}{2F_c^2}\left[{}_oS - S_f\right]$$

Adding $\frac{-1}{{}_oS}$ and multiplying by $\frac{-A_2|{}_o\dot{C}|}{A_6({}_oS - S_f)}$, this becomes

$$\frac{A_2|{}_o\dot{C}|}{A_6S_f({}_oS - S_f)} - \frac{A_2A_6|{}_o\dot{C}|\tau_1^2\delta_{\underline{r}}^2}{2F_c^2}$$

$$= \frac{A_2|{}_o\dot{C}|}{A_6S_f({}_oS - S_f)} - \frac{A_6({}_oS - S_f)^2\tau_1^2}{2A_2|{}_o\dot{C}|}$$

by using equations 14 and 16 .

Therefore the integral of $\dot{C}$ is

$$\lim_{x \to \infty} \int_o^x \dot{C}\,dt = {}_o\dot{C}x - {}_o\dot{C}(S_f - {}_oS) \Big/ {}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)$$

$$- \frac{{}_o\dot{C}|{}_o\dot{C}|A_{2o}S}{A_6S_f^2({}_oS-S_f)} + \frac{A_6({}_oS - S_f)^2{}_o\dot{C}S_f}{2A_{2o}S^3|{}_o\dot{C}|A_1^2\left(\frac{{}_oS}{S_f} + A_3 + 1\right)^2}$$

Multiplying both sides of equation 21 by S_f/S_{n-1},

$$\frac{A_6({}_oS - S_f)}{A_2|{}_o\dot{C}|} = \frac{(A_4S_f + A_5|{}_o\dot{C}|)}{S_f}$$

Therefore,

$$\lim_{x \to \infty} \int_o^x \dot{C}\,dt = {}_o\dot{C}\left[x + ({}_oS - S_f) \Big/ {}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right) - {}_oS/S_f(A_4S_f + A_5|{}_o\dot{C}|) + \frac{({}_oS - S_f)(A_4S_f + A_5|{}_o\dot{C}|)}{2{}_oS^3A_1^2\left(\frac{{}_oS}{S_f} + A_3 + 1\right)^2}\right] \quad (47$$

This equation can also be written in the following form if $\delta_{\underline{r}}$ is an input rather than ${}_o\dot{C}$.

$$\lim_{x \to \infty} \int_o^x \dot{C}\,dt = \frac{{}_o\dot{C}}{{}_oS}\left[{}_oSx - \frac{2(1 + A_2|\delta_{\underline{r}}|)^2}{A_4 + \sqrt{A_4^2 + 4A_5A_6|\delta_{\underline{r}}|}} + \right.$$

$$\frac{A_2|\delta_r|}{(1 + A_2|\delta_r|)A_1(2 + A_2|\delta_r| + A_3)} + \frac{A_2|\delta_r|\left(A_4 + \sqrt{A_4^2 + 4A_5A_6|\delta_r|}\right)}{4(1 + A_2|\delta_r|)^2A_1^2(2 + A_2|\delta_r| + A_3)^2}\Bigg]$$

from equations 7 and 8 .

A.1.3.2 Instructor Control

In the instructor control situation, $\dot{C}$ increases directly with time until it equals ${}_o\dot{C}$, at which time it is set equal to ${}_o\dot{C}$. It is kept at ${}_o\dot{C}$ for the remainder of the turn. The growth rate is determined as follows.

$$\int_o^{{}_o\dot{C}/m} mt\,dt + \lim_{x\to\infty} \int_{{}_o\dot{C}/m}^{x} {}_o\dot{C}\,dt = \lim_{x\to\infty} \int_o^{x} \dot{C}\,dt$$

from equation 47 ,

$$\left(\frac{{}_o\dot{C}}{m}\right)^2\frac{m}{2} + {}_o\dot{C}\left(x - \frac{{}_o\dot{C}}{m}\right) = {}_o\dot{C}\left[x + \frac{({}_oS - S_f)}{{}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)} - \frac{{}_oS}{S_f(A_4S_f + A_5|{}_o\dot{C}|)} + \frac{({}_oS - S_f)(A_4S_f + A_5|{}_o\dot{C}|)}{2{}_oS^3A_1^2\left(\frac{{}_oS}{S_f} + A_3 + 1\right)^2}\right]$$

$$m = \frac{{}_o\dot{C}}{2}\left[\frac{{}_oS}{S_f(A_4S_f + A_5|{}_o\dot{C}|)} - \frac{({}_oS - S_f)}{{}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)}\left(1 + \frac{A_4S_f + A_5|{}_o\dot{C}|}{2{}_oSA_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)}\right)\right]^{-1} \tag{48}$$

Therefore,

$${}_o\dot{C}_n = 0 \text{ when } nh < T/2$$

$$\dot{C}_n = \dot{C}_{n-1} + \frac{h\,{}_o\dot{C}}{2}\Bigg/\left[\frac{{}_oS}{S_f(A_4S_f + A_5|{}_o\dot{C}|)} - \frac{({}_oS - S_f)}{{}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)}\left(1 + \frac{A_4S_f + A_5|{}_o\dot{C}|}{2{}_oSA_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)}\right)\right] \tag{49}$$

when $nh > \frac{T}{2}$ and $|\dot{C}_n| < |{}_o\dot{C}|$

$\dot{C}_n = {}_o\dot{C}$ otherwise.

A. 1. 3. 3 Program Control

In the program control situation $\dot{C}$ changes abruptly from zero to ${}_o\dot{C}$ after an appropriate time delay. The time delay can be determined as follows.

$$ {}_o\dot{C}(x - \tau) = \lim_{x \to \infty} \int_o^x \dot{C}\, dt $$

from equation 47 ,

$$ \tau = \frac{{}_oS}{S_f(A_4 S_f + A_5 |{}_o\dot{C}|)} - \frac{({}_oS - S_f)}{{}_oS^2 A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)} \left(1 + \frac{A_4 S_f + A_5 |{}_o\dot{C}|}{2\,{}_oSA_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)}\right) \tag{50} $$

Therefore,

$$ \dot{C}_n = 0 \text{ when } nh < \tau $$

$$ \dot{C}_n = {}_o\dot{C} \text{ when } nh > \tau \tag{51} $$

A. 1. 4 Submarine Dive

A. 1. 4. 1 Solution of the Differential Equation

Equation 3 is the differential equation describing the dive angle, (D), as a function of stern plane deflection angle, δ_s. S, the speed of the submarine, is assumed to remain constant throughout any dive maneuver.

$$\ddot{D} = -(A_7 S\dot{D} + A_8\dot{D}\,|\dot{D}| + A_9 D + A_{11}S^2\delta_s).$$

When $\dot{D}$ is positive this becomes

$$\ddot{D} = -(A_7 S\dot{D} + A_8\dot{D}^2 + A_9 D + A_{11}S^2\delta_s).$$

While for $\dot{D}$ negative it is

$$\ddot{D} = -(A_7 S\dot{D} - A_8\dot{D}^2 + A_9 D + A_{11}S^2\delta_s).$$

So the graph of $\ddot{D}$ versus $\dot{D}$ is two truncated parabolas, joined at point 0, $-(A_9 D + A_{11}S^2\delta_s)$. The slope of the graph at that point is $-A_7 S$. A_7, A_9 and A_{11} are positive while A_8 will always be negative. Thus, the slope of the graph near $\dot{D} = 0$ will be negative. This is shown in Figure A-2.

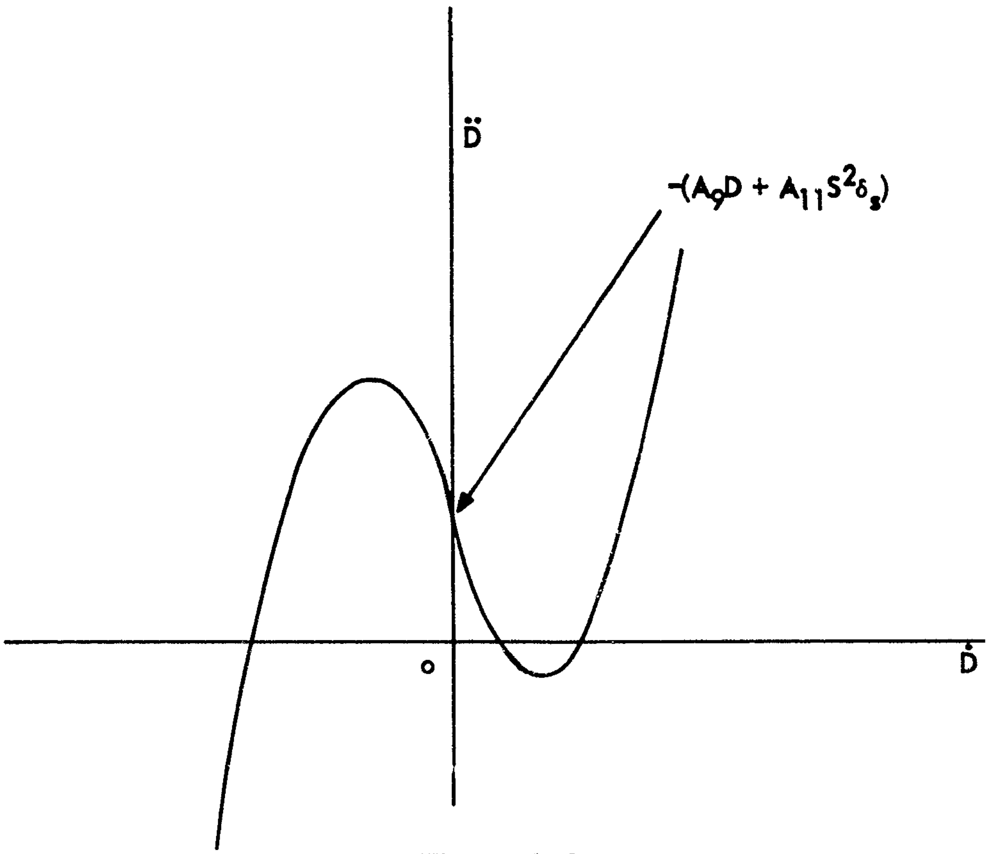

Figure A-2.

The slope becomes positive when $|\dot{D}|$ is large enough. The area of interest for simulator purposes however, is that where the slope of $\ddot{D}$ versus $\dot{D}$ is negative.

Initially, $\dot{D}$ and D are both zero; therefore $\ddot{D} = A_{11}S^2\delta_s$. Consider then, what would happen if an initial negative δ_s was applied. As stated previously, the corresponding initial $\ddot{D}$ is $-A_{11}S^2\delta_s$, which is a positive quantity. Thus, $\dot{D}$ would increase from its initial value of zero. This increase would result in two simultaneous reactions. First, the point on the graph of $\ddot{D}$ versus $\dot{D}$ representing the present state of the system would begin to move in the direction of increasing $\dot{D}$. Since the graph has a negative slope, the $\ddot{D}$ coordinate of the point would decrease. That is, $\dot{D}$ would be made to increase at a slower rate until the current-system-state point reached the intersection of the curve and the $\dot{D}$ axis. At this point, $\ddot{D} = 0$ and $\dot{D}$ will remain fixed.

On the other hand, consider what would happen if the minimum value of $\ddot{D}$ for $\dot{D} > 0$ was not less than zero (refer to Figure A-2). As $\dot{D}$ continued to increase, $\ddot{D}$, although decreasing, would remain greater than zero. Thus, $\dot{D}$ would continue to increase until the current-system-state point reached that portion of the graph where the slope is positive. This represents a completely unstable state, one which the design of the equations must prevent under all normal conditions.

The second reaction to the initial negative δ_s would be that D would also increase. Since the $\ddot{D}$ intercept is $-(A_9D + A_{11}S^2\delta_s)$, the increase of D would tend to negate the original negative δ_s causing the $\ddot{D}$ intercept to move to zero. Thus, $\ddot{D}$ would decrease more rapidly than it would if affected only by the negative slope of the graph. In effect, the $\dot{D}$ axis would move up, towards the $\ddot{D}$ intercept with rate A_9D, while the curve remained fixed. The rapidity with which this occurs will determine the time it takes for $\dot{D}$ to go to zero, fixing the dive angle at that value corresponding to δ_s. A_9 is small however so this occurs at a very slow rate. As an example, assume that $\delta_s = -10°$ and S = 10 knots, $\dot{D}$ would not go to zero until D = 41°, which is a very steep dive angle.

Figure A-2 is a graph of $\ddot{D}$ versus $\dot{D}$. The maximum is at $\dot{D} = \frac{A_7S}{2A_8}$ while the minimum is at $\dot{D} = -\frac{A_7S}{A_8}$. Since A_8 is negative, the maximum will be in the $\dot{D} < 0$ portion of the graph and the minimum in the $\dot{D} > 0$ portion for a negative slope between the maximum and minimum points.

Another requirement on the points is that they should be above and below $\dot{D}$ axis respectively. Without this requirement the possibility exists of the current-system-state point reaching the positive-slope portion of the graph, as described previously. The

requirement that the $\dot{D}$ axis separate the maximum and minimum points on the graph leads to the following requirements for the constants.

$$\left| A_9 D + A_{11} S^2 \delta_s \right| < - \frac{(A_7 S)^2}{4A_8} \tag{52}$$

This is found by substituting $\frac{A_7 S}{2A_8}$ and $-\frac{A_7 S}{2A_8}$ for $\dot{D}$ in equation 3, and is useful as a quick check of a set of constants (refer to section 3.5).

In order to proceed with the solution of the differential equation it will be necessary to approximate the portion of the graph of interest with a straight line. That portion lies between $\frac{A_7 S}{2A_8}$ and $-\frac{A_7 S}{2A_8}$. A least squares approximation will be used to find the line. The square of the error will be integrated between $\frac{A_7 S}{2A_8}$ and $-\frac{A_7 S}{2A_8}$.

The line will intersect point $(0, - A_9 D + A_{11} S^2 \delta_s)$, as does the graph. Since the graph is symmetric about that point, the least squares integral will be taken from $\dot{D} = 0$ to $\dot{D} = -\frac{A_7 S}{2A_8}$. Let -a be the slope of the line.

$$\int_0^{-\frac{A_7 S}{2A_8}} [-(A_7 S\dot{D} + A_8 \dot{D}^2 + A_9 D + A_{11} S^2 \delta_s) - (-a\dot{D} - A_9 D - A_{11} S^2 \delta_s)]^2 \, d\dot{D}$$

$$= \int_0^{-\frac{A_7 S}{2A_8}} [(a - A_7 S)\dot{D} - A_8 \dot{D}^2]^2 \, d\dot{D} = \left[(a - A_7 S)^2 \frac{\dot{D}^3}{3} - 2A_8 (a - A_7 S) \frac{\dot{D}^4}{4} + A_8^2 \frac{\dot{D}^5}{5} \right]_0^{-\frac{A_7 S}{2A_8}}$$

This must be minimized as a function of a variable a.

Differentiating with respect to a,

$$\frac{2}{3}(a - A_7S)\left(-\frac{A_7S}{2A_8}\right)^3 - \frac{2}{4}A_8\left(-\frac{A_7S}{2A_8}\right)^4 = 0$$

$$a - A_7S = -\frac{3}{4}\frac{A_7S}{2} \qquad (5$$

$$a = \frac{5}{8}A_7S$$

So the equation

$$\ddot{D} = -\frac{5}{8}A_7S\dot{D} - (A_9D + A_{11}S^2\delta_s) \qquad (5$$

will be used to approximate equation 3.

Equation 54 is a linear, second order differential equation. The forcing function i δ_s. δ_s will be assumed a linear function of time.

$$\delta_s = \delta_1 + \dot{\delta}_s t \qquad (5$$

The homogeneous equation associated with equation 54 is (using $a = \frac{5}{8}A_7S$),

$$\ddot{D} + a\dot{D} + A_9D = 0 \qquad (5$$

Equation 56 has the solution

$$D = Ae^{(-a + \sqrt{a^2 - 4A_9})\,t/2} + Be^{(-a - \sqrt{a^2 - 4A_9})\,t/2} \qquad (5$$

The non-homogeneous equation is

$$\ddot{D} + a\dot{D} + A_9D + A_{11}S^2(\delta_1 + \dot{\delta}_s t) = 0 \qquad (5$$

Consider the particular solution

$$D = D_1 + D_2 t \qquad (5$$

Putting this into equation 58,

$$0 + aD_2 + A_9(D_1 + D_2t) + A_{11}S^2(\delta_1 + \dot{\delta}_s t) = 0$$

This equation has to be valid for all values of t in order for equation 59 to be a solution

$$aD_2 + A_9D_1 + A_{11}S^2\delta_1 = 0 \qquad \text{for } t = o$$

$$A_9D_2 + A_{11}S^2\dot{\delta}_s = 0 \qquad \text{for } t = 1 \text{ with above substitution}$$

Solving,

$$D_2 = - A_{11}S^2\dot{\delta}_s / A_9$$

$$D_1 = \frac{aA_{11}S^2\dot{\delta}_s}{A_9^2} - \frac{A_{11}S^2\delta_1}{A_9} \qquad (60)$$

$$D = \frac{A_{11}S^2}{A_9^2}\left[a\dot{\delta}_s - A_9\delta_1 - A_9\dot{\delta}_s t\right]$$

$$D = \frac{A_{11}S^2}{A_9^2}\left[a\dot{\delta}_s - A_9\delta_s\right]$$

The general solution is obtained by adding equations 57 and 60.

$$D = Ae^{\left(-a+\sqrt{a^2-4A_9}\right)t/2} + Be^{\left(-a-\sqrt{a^2-4A_9}\right)t/2} + \frac{A_{11}S^2}{A_9^2}\left(a\dot{\delta}_s - A_9\delta_s\right) \qquad (61)$$

A.1.4.2 Four Control Situations

Operator Control

Since there are two constants in equation 61 , two past values of D must be used tc evaluate them. δ_{s_n} is an input from the operator.

Let

$$x_1 = (-a + \sqrt{a^2 - 4A_9})/2$$

$$x_2 = (-a - \sqrt{a^2 - 4A_9})/2 \tag{62}$$

$$-D_n + \left(Ae^{x_1(n-1)h}\right)e^{x_1 h} + \left(Be^{x_2(n-1)h}\right)e^{x_2 h} = -\frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_n})$$

$$D_{n-1} = \left(Ae^{x_1(n-1)h}\right) + \left(Be^{x_2(n-1)h}\right) + \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-1}})$$

$$D_{n-2} = \left(Ae^{x_1(n-1)h}\right)e^{-x_1 h} + \left(Be^{x_2(n-1)h}\right)e^{-x_2 h} + \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-2}})$$

There are three unknowns, D_n, $\left(Ae^{x_1(n-1)h}\right)$ and $\left(Be^{x_1(n-1)h}\right)$ The denominator determinant is

$$D = \begin{vmatrix} -1 & e^{x_1 h} & e^{x_2 h} \\ 0 & 1 & 1 \\ 0 & e^{-x_1 h} & e^{-x_2 h} \end{vmatrix}$$

$$D = e^{-x_1 h} - e^{-x_2 h}$$

$$D_nD = \begin{vmatrix} -\frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_n}) & e^{x_1h} & e^{x_2h} \\ D_{n-1} - \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-1}}) & 1 & 1 \\ D_{n-2} - \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-2}}) & e^{-x_1h} & e^{-x_2h} \end{vmatrix}$$

$$\begin{aligned} D_nD = {} & \left(e^{-x_1h} - e^{-x_2h}\right)\frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_n}) \\ & -\left(D_{n-1} - \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-1}})\right)\left(e^{(x_1-x_2)h} - e^{(x_2-x_1)h}\right) \\ & +\left(D_{n-2} - \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-2}})\right)\left(e^{x_1h} - e^{x_2h}\right) \end{aligned} \tag{63}$$

Where,

$$\begin{aligned} \frac{e^{(x_1 - x_2)h} - e^{(x_2 - x_1)h}}{e^{-x_1h} - e^{-x_2h}} &= e^{(x_1+x_2)h}\frac{\left(e^{-2x_2h} - e^{-2x_1h}\right)}{e^{-x_1h} - e^{-x_2h}} \\ &= -e^{(x_1+x_2)h}\left(e^{-x_1h} + e^{-x_2h}\right) \\ &= -\left(e^{x_2h} + e^{x_1h}\right) \end{aligned}$$

While,

$$\frac{e^{x_1h} - e^{x_2h}}{e^{-x_1h} - e^{-x_2h}} = \frac{e^{x_1h} - e^{x_2h}}{\frac{e^{x_2h} - e^{x_1h}}{e^{(x_1+x_2)h}}} = -e^{(x_1+x_2)h}$$

So equation 62 becomes

$$D_n = \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_n}) + \left(D_{n-1} - \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-1}})\right)(e^{x_1h} + e^{x_2h})$$

$$-\left(D_{n-2} - \frac{A_{11}S^2}{A_9^2}(a\dot{\delta}_s - A_9\delta_{s_{n-2}})\right)e^{(x_1+x_2)h} \tag{64}$$

Any simplification of equation 64 involves expansion of the exponential. This depends on the size of x_1h and x_2h. Since a is positive, equation 53, and A_9 is positive, x_2h has the larger magnitude of the two. The magnitude of x_2h is less than ah. The values given for A_7 in the FBM Math Model are far enough apart to cast considerable doubt as to the possibility of any realistic statements about A_7.* If the corrected value is used for the second submarine then the expansion of the exponential in equation 64 requires that, for the first submarine, hS be less than 64 yards or 114 knot-seconds, while for the second it be less than 107 yards or 189 knot-seconds.

The extent to which these requirements are met determines the reliability of the following expansion. First equation 64 will be rewritten with some of the terms rearrang

$$D_n - D_{n-1} = (D_{n-1} - D_{n-2})e^{(x_1+x_2)h} - D_{n-1}\left(1 + e^{(x_1+x_2)h} - [e^{x_1h} + e^{x_2h}]\right)$$

$$+ \frac{A_{11}S^2a}{A_9^2}\left(1 + e^{(x_1+x_2)h} - [e^{x_1h} + e^{x_2h}]\right)\dot{\delta}_s$$

$$- \frac{A_{11}S^2}{A_9}\left(\delta_{s_n} - \delta_{s_{n-1}}[e^{x_1h} + e^{x_2h}] + \delta_{s_{n-2}}e^{(x_1+x_2)h}\right)$$

The number of terms that will be retained in the exponential expansion depends on where it appears in the equation. Enough terms will be kept so that none of the variables will have a coefficient independent of h. This will become clear as the analysis proceeds. The coefficient of $D_{n-1} - D_{n-2}$ is $1 + (x_1 + x_2)h = 1 - ah$. The coefficient of $-D_{n-1}$ is

*Using the test given by equation 52 only one of them could possibly be correct. It was possible, however, to correct one of the others using tactical trial data. The third could not be used at all.

$$1 + [1 + (x_1 + x_2)h + (x_1 + x_2)^2 \frac{h^2}{2}] - [1 + x_1 h + x_1{}^2 \frac{h^2}{2} + 1 + x_2 h + x_2{}^2 \frac{h^2}{2}]$$

$$= x_1 x_2 h^2 = A_9 h^2 \tag{65}$$

The next two terms of the equation must be added together, since $\delta_{s_n} - \delta_{s_{n-1}} = \delta_{s_{n-1}} - \delta_{s_{n-2}} = \dot{\delta}_s h$. The coefficient of $\frac{A_{11} S^2}{A_9}$ is

$$\frac{a\dot{\delta}_s}{A_9}\Bigg[1 + 1 + (x_1 + x_2)h + (x_1 + x_2)^2 \frac{h^2}{2} + (x_1 + x_2)^3 \frac{h^3}{6}$$

$$- \left(1 + x_1 h + x_1{}^2 \frac{h^2}{2} + x_1{}^3 \frac{h^3}{6} + 1 + x_2 h + x_2{}^2 \frac{h^2}{2} + x_2{}^3 \frac{h^3}{6}\right)\Bigg]$$

$$-\Bigg[\delta_{s_n} - \delta_{s_{n-1}}\left(1 + x_1 h + x_1{}^2 \frac{h^2}{2} + 1 + x_2 h + x_2{}^2 \frac{h^2}{2}\right)$$

$$+ \delta_{s_{n-2}}\left(1 + (x_1 + x_2)h + (x_1 + x_2)^2 \frac{h^2}{2}\right)\Bigg] \tag{66}$$

Since $\delta_{s_n} - 2\delta_{s_{n-1}} + \delta_{s_{n-2}} = 0$, this is equal to

$$\frac{a\dot{\delta}_s}{A_9}[x_1 x_2 h^2 + x_1 x_2 (x_1 + x_2)\frac{h^3}{2}]$$

$$+\left(\delta_{s_{n-1}} - \delta_{s_{n-2}}\right) \left(x_1 + x_2)h + (x_1{}^2 + x_2{}^2) \frac{h^2}{2}\right) - \delta_{s_{n-2}} x_1 x_2 h^2$$

however, $x_1 x_2 = A_9$ so $\frac{a x_1 x_2}{A_9} = a$. Furthermore,

$$\left(\delta_{s_{n-1}} - \delta_{s_{n-2}}\right)(x_1 + x_2)h = \dot{\delta}_s (x_1 + x_2)h^2 = -a\dot{\delta}_s h^2.$$ So

$$\frac{a\dot{\delta}_s}{A_9} x_1 x_2 h^2 + \left(\delta_{s_{n-1}} - \delta_{s_{n-2}}\right)(x_1 + x_2)h = 0$$

Similarly, $\frac{a\dot{\delta}_s}{A_9} x_1 x_2 (x_1 + x_2) \frac{h^3}{2} = -a^2 \dot{\delta}_s \frac{h^3}{2}$

while $\left(\delta_{s_{n-1}} - \delta_{s_{n-2}}\right)(x_1{}^2 + x_2{}^2) \frac{h^2}{2} = \dot{\delta}_s \frac{h^3}{2}(x_1^2 + x_2^2) = (a^2 - 2A_9)\dot{\delta}_s \frac{h^3}{2}$

So expression 66 is equal to

$$-A_9 \dot{\delta}_s h^3 - \delta_{s_{n-2}} A_9 h^2 = -A_9\left(\delta_{s_{n-2}} + (\delta_{s_{n-1}} - \delta_{s_{n-2}})\right)h^2 = -A_9 \delta_{s_{n-1}} h^2$$

Combining this with equations 53 and 65 , equation 64 can now be written

$$D_n - D_{n-1} = (D_{n-1} - D_{n-2})(1 - \frac{5}{8} A_7 Sh) - A_9 D_{n-1} h^2 - A_{11} S^2 h^2 \delta_{s_{n-1}} \quad (67$$

Command Control

The command control situation will be handled differently from the way the same situatiun wa ndled in the speed and turn maneuvers. There are two commands that mig be given, ordered D or ordered $\dot{D}$. Ordered D cannot be handled directly. If $\ddot{D}$ and $\dot{D}$ go t zero then $A_9 D + A_{11} S^2 \delta_s = 0$. Using a magnitude of 0.0001 for A_{11} and 0.001 for A_9 (both of which are very reliable estimates), δ_s corresponding to ${}_oD$ is $\frac{-10\,{}_oD}{S^2}$.

For an order dive angle of 30° at a speed of 20 knots δ_s would be approximately 2.4 This leads to a $\dot{D}$ that is much too slow.

Therefore, instead of an ordered D, the command control situation will be given i terms of ${}_o\dot{D}$. This will be a more flexible formulation since each ${}_o\dot{D}$ corresponds to a par ticular value of δ_s. ${}_o\dot{D}$ is achieved when $\ddot{D} = 0$. Once this state is reached, $\dot{D}$ will start to decrease toward zero if δ_s is held constant, however, this happens very slowly. The effec of $A_9 D$ is so insignificant that it will be ignored in the command control situation. The difference between the command control and operator control situation is just a deletion of the effect of $A_9 D$. The $A_9 D$ term is not completely ignored, however. When there is one valu of D which is most important in an overshoot or level-up maneuver where D varies around

some central, non-zero value, D will be denoted by ${}_oD$ and used in finding the relation between ${}_o\dot{D}$ and δ_s. This value of D will be used in the following calculations. Bearing in mind, however, that it will usually be set to zero. The error introduced by ignoring it completely will always be less than 10%.

Consider equation 54 with a fixed value of δ_s. δ_s corresponds to ${}_o\dot{D}$ in the followi equation, derived by setting $\ddot{D} = 0$ in equation 54.

$$\delta_s = -(\frac{5}{8} A_7 S_o\dot{D} + A_{9o}D)\frac{1}{A_{11}S^2} \tag{68}$$

where ${}_oD$, as described above, is often zero.

Equation 54 becomes

$$\ddot{D} = -\frac{5}{8} A_7 S\dot{D} - (A_{9o}D - [A_{9o}D + \frac{5}{8} A_7 S_o\dot{D}])$$

$$\ddot{D} = -\frac{5}{8} A_7 S(\dot{D} - {}_o\dot{D}) \tag{69}$$

Equation 69 is a linear, first order differential equation in $\dot{D}$. Its solution is

$$\dot{D} - {}_o\dot{D} = Ae^{-\frac{5}{8} A_7 St} \tag{70}$$

$$\dot{D}_n = Ae^{-\frac{5}{8} A_7 Snh} + {}_o\dot{D}$$

$$\dot{D}_{n-1} = \left(Ae^{-\frac{5}{8} A_7 Snh}\right)e^{+\frac{5}{8} A_7 Sh} + {}_o\dot{D}$$

$$\dot{D}_n - {}_o\dot{D} = (\dot{D}_{n-1} - {}_o\dot{D})e^{-\frac{5}{8} A_7 Sh} \tag{71}$$

Equation 71 may either be simplified, using the arguments following equation 64 or used as it is.

Simplifying,

$$\dot{D}_n - {}_o\dot{D} = (\dot{D}_{n-1} - {}_o\dot{D})(1 - \tfrac{5}{8} A_7 S h)$$

$$\dot{D}_n - \dot{D}_{n-1} = ({}_o\dot{D} - \dot{D}_{n-1})\tfrac{5}{8} A_7 S h \tag{72}$$

Note also that equation 72 , used together with equation 68 , will provide a less accurate but simpler operator control equation.

$$\dot{D}_n - \dot{D}_{n-1} = -(A_{9o} D + A_{11} S^2 \delta_s + \tfrac{5}{8} A_7 S \dot{D}_{n-1})h \tag{73}$$

When equation 72 or 73 is used to update $\dot{D}$, then D is updated using the formula

$$D_n = D_{n-1} + \tfrac{h}{2}(\dot{D}_n + \dot{D}_{n-1}) \tag{74}$$

<u>Instructor Control</u>

The instructor control situation is simulated by an abrupt change in $\dot{D}_n$ after an appropriate time delay. $\dot{D}_n$ changes from its original value to ${}_o\dot{D}$ and is held until $D = {}_oD$, at which time it is set to zero. The time delay is found by equating the D at some future time (x) found by using equation 70 to the value of D at that time found by using the instructor control formulation.

In equation (70), let $\dot{D} = \dot{D}_o$ when $t = 0$.

So $$\lim_{x\to\infty} \int_0^x \dot{D}\, dt = \lim_{x\to\infty} \int_0^x \left[{}_o\dot{D} + (\dot{D}_o - {}_o\dot{D}) e^{-\frac{5}{8} A_7 S t}\right] dt$$

$$= \lim_{x\to\infty} \left[{}_o\dot{D}x - \frac{8(\dot{D}_o - {}_o\dot{D})}{5 A_7 S}\left(e^{-\frac{5}{8} A_7 S x} - 1\right)\right]$$

$$= {}_o\dot{D}x + \frac{8(\dot{D}_o - {}_o\dot{D})}{5 A_7 S} \tag{75}$$

But this is to be equated to

$$\int_0^\tau \dot{D}_o\, dt + \lim_{x\to\infty} \int_\tau^x {}_o\dot{D}\, dt = (\dot{D}_o - {}_o\dot{D})\tau + {}_o\dot{D}x$$

therefore

$$\tau = \frac{8}{5A_7S} \tag{76}$$

$$\dot{D}_n = \dot{D}_o \quad \text{when } nh \le \frac{8}{5A_7S}$$

$$\dot{D}_n = {}_o\dot{D} \quad \text{when } nh > \frac{8}{5A_7S} \text{ and } ({}_oD - D_n)({}_o\dot{D}) > 0$$

$$\text{where } D_n = D_{n-1} + \frac{h}{2}(\dot{D}_n + \dot{D}_{n-1})$$

$$\dot{D}_n = 0 \text{ otherwise} \tag{77}$$

Program Control

In the program control situation, the inputs are D_o, ${}_oD$, $\dot{D}_o$ and ${}_o\dot{D}$. D_n changes abruptly from D_o to ${}_oD$ in such a way that the depth, $dZ = -S \sin D\, dt$, will be the same at some future time (x) as it would have been using the formulation described in equation 77 .

Consider the formulation set forth in equation 77 with $D = D_o$ when $t = 0$.

$D = D_o + \dot{D}_o t$ when $0 \le t \le \frac{8}{5}A_7S$.

When $t = \frac{8}{5A_7S}$, $D = D_o + \frac{8\dot{D}_o}{5A_7S}$. $\dot{D} = {}_o\dot{D}$ when $t > \frac{8}{5A_7S}$ but $D < {}_oD$. When $D = {}_oD$, $\dot{D}$ goes to zero. The time at which this occurs can be calculated as follows: For $t \ge \frac{8}{5A_7S}$,

$D = D_o + \frac{8\dot{D}_o}{5A_7S} + {}_o\dot{D}(t - \frac{8}{5A_7S})$. Equating this to ${}_oD$ produces

$$t = \frac{1}{{}_o\dot{D}}\left[{}_oD - D_o + \frac{8({}_o\dot{D} - \dot{D}_o)}{5A_7S}\right]$$

as the time at which D first becomes ${}_oD$. D is then fixed at ${}_oD$. The complete formulation is therefore

$$D = D_o + \dot{D}_o t \text{ when } 0 \le t \le \frac{8}{5A_7S}$$

$$D = D_o + \frac{8\dot{D}_o}{5A_7S} + {}_o\dot{D}\left(t - \frac{8}{5A_7S}\right)$$

$$\text{when } \frac{8}{5A_7S} \le t \le \frac{1}{{}_o\dot{D}}\left[{}_oD - D_o + \frac{8({}_o\dot{D} - \dot{D}_o)}{5A_7S}\right]$$

$$D = {}_oD \qquad \text{when } \frac{1}{{}_o\dot{D}}\left[{}_oD - D_o + \frac{8({}_o\dot{D} - \dot{D}_o)}{5A_7S}\right] \le t$$

Let $t_1 = \frac{8}{5} A_7 S$ and $t_2 = \frac{1}{{}_o\dot{D}}\left[{}_oD - D_o + \frac{8({}_o\dot{D} - \dot{D}_o)}{5A_7S}\right]$

therefore, when $t = x$,

$$Z = Z_o - S\int_0^{t_1} \sin(D_o + \dot{D}_o t)\,dt - S\int_{t_1}^{t_2} \sin[D_o + t_1(\dot{D}_o - {}_o\dot{D}) + {}_o\dot{D}t]\,dt$$

$$-S\int_{t_2}^{x} \sin {}_oD\,dt \tag{78}$$

$$Z - Z_o + Sx \sin {}_oD = S\left\{\frac{1}{\dot{D}_o}[\cos(D_o + \dot{D}_o t_1) - \cos D_o] + t_2 \sin {}_oD\right.$$

$$+ \frac{1}{{}_o\dot{D}}[\cos(D_o + t_1(\dot{D}_o - {}_o\dot{D}) + {}_o\dot{D}t_2)$$

$$\left. - \cos(D_o + t_1(\dot{D}_o - {}_o\dot{D}) + {}_o\dot{D}t_1)]\right\}$$

where $D_o + t_1(\dot{D}_o - {}_o\dot{D}) + {}_o\dot{D}t_2 = {}_oD$

$$Z - Z_o + Sx \sin {}_oD = S\left\{\left(\frac{1}{\dot{D}_o} - \frac{1}{{}_o\dot{D}}\right)\cos(D_o + \dot{D}_o t_1) + t_2 \sin {}_oD\right.$$

$$\left. + \frac{1}{{}_o\dot{D}} \cos {}_oD - \frac{1}{\dot{D}_o} \cos D_o\right\} \tag{79}$$

On the other hand, the program control formulation leads to

$$Z = Z_o - S\int_0^{\tau} \sin D_o\,dt - S\int_{\tau}^{x} \sin {}_oD\,dt$$

$$Z - Z_o + Sx \sin {}_oD = S\tau(\sin {}_oD - \sin D_o) \tag{80}$$

therefore, equating the right hand members of equations 79 and 80,

$$\tau(\sin {}_oD - \sin D_o) = \frac{1}{\dot{D}_o}\left[\cos\left(D_o + \frac{8\dot{D}_o}{5A_7S}\right) - \cos D_o\right]$$

$$+ \frac{1}{{}_o\dot{D}}\left[({}_oD - D_o)\sin {}_oD + \frac{8({}_o\dot{D} - \dot{D}_o)}{5A_7S} \sin {}_oD - \cos\left(D_o + \frac{8\dot{D}_o}{5A_7S}\right) + \cos {}_oD\right] \quad (8$$

where all angles are expressed in radians.

This formula more than satisfies the requirements for most maneuvers for whicl program control would be used. Another form that could be used is that which is used tc change from ${}_oD$ to D_o using ${}_o\dot{D}$, where $\dot{D}_o = 0$. When $\dot{D}_o \to 0$, the $\frac{1}{\dot{D}_o}$ term $\to -\frac{8}{5A_7S}$ sin I Therefore, equation 81 can be written

$$\tau(\sin {}_oD - \sin D_o) = \frac{8}{5A_7S}(\sin {}_oD - \sin D_o) + \frac{1}{{}_o\dot{D}}[({}_oD - D_o)\sin {}_oD + \cos {}_oD - \cos \text{I}$$

$$\tau = \frac{8}{5A_7S} + \frac{1}{{}_o\dot{D}}\left[\frac{({}_oD - D_o)\sin {}_oD + \cos {}_oD - \cos D_o}{\sin {}_oD - \sin D_o}\right] \quad ($$

Consequently, when $D_o = 0$,

$$\tau = \frac{8}{5A_7S} + \frac{{}_oD}{{}_o\dot{D}} - \frac{(1 - \cos {}_oD)}{{}_o\dot{D} \sin {}_oD} \quad ($$

In any case, the program control formulation is

$D_n = D_o$ when $nh \leq \tau$

$D_n = {}_oD$ when $nh > \tau$ (

A.1.5 Sine and Cosine Approximation

In most of the submarine formulations presented above, the output is in the form an angle which represents the orientation of the submarine with respect to a particular : erence axes. These angles, in most formulations, are used to update the submarine po

If sine or cosine subroutines have to be used to achieve this, then much of the computer time saved by all these simplifications will be lost again. Therefore, it is necessary to develop some approximation formulas.

Suppose the output is in the form of an angle rate of change and a speed. $\dot{C}_n$ and S

$$y_n - y_{n-1} = \int_{(n-1)h}^{nh} S \cos C \, dt = \frac{(S_n + S_{n-1})}{2} \int_0^h \cos(C_{n-1} + \left[\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right] t) dt$$

$$y_n - y_{n-1} = \frac{(S_n + S_{n-1})}{2} \left[\sin\left(C_{n-1} + \left[\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right] h\right) - \sin C_{n-1}\right] \frac{2}{(\dot{C}_n + \dot{C}_{n-1})}$$

and similarly

$$x_n - x_{n-1} = \frac{(S_n + S_{n-1})}{2} \left[\cos C_{n-1} - \cos\left(C_{n-1} + \left[\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right] h\right)\right] \frac{2}{(\dot{C}_n + \dot{C}_{n-1}}$$

However, $(\frac{\dot{C}_n + \dot{C}_{n-1}}{2})h$ will always be small enough ($|\dot{C}|$ very rarely > 5° /sec) so that the approximations $\sin\left[\left(\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right) h\right] = \left(\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right) h$

and

$$\cos\left[\left(\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right) h\right] = 1 - \left(\frac{\dot{C}_n + \dot{C}_{n-1}}{2}\right)^2 \frac{h^2}{2}$$

will lead to an error considerably less than 10%.

Using these approximations, let

$$\Delta = \frac{\dot{C}_n + \dot{C}_{n-1}}{2} \; h$$

Therefore

$$\frac{h}{\Delta} [\sin(C_{n-1} + \Delta) - \sin C_{n-1}] = \frac{h}{\Delta} [(\sin C_{n-1})\left(1 - \frac{\Delta^2}{2}\right) + \Delta \cos C_{n-1} - \sin C_{n-1}]$$

$$= h [\cos C_{n-1} - \frac{\Delta}{2} \sin C_{n-1}]$$

$$\frac{h}{\Delta} [\cos C_{n-1} - \cos(C_{n-1} + \Delta)] = \frac{h}{\Delta}\left[\cos C_{n-1} - (\cos C_{n-1})\left(1 - \frac{\Delta^2}{2}\right) + \Delta \sin C_{n-1}\right]$$

$$= h[\sin C_{n-1} + \frac{\Delta}{2} \cos C_{n-1}] \quad (85$$

Therefore

$$y_n - y_{n-1} = (S_n + S_{n-1})\frac{h}{2}[\cos C_{n-1} - (\dot{C}_n + \dot{C}_{n-1})\frac{h}{4} \sin C_{n-1}]$$

and

$$x_n - x_{n-1} = (S_n + S_{n-1})\frac{h}{2}[\sin C_{n-1} + (\dot{C}_n + \dot{C}_{n-1})\frac{h}{4} \cos C_{n-1}]$$

where $\dot{C}_n$ and $\dot{C}_{n-1}$ are expressed in radians per second.

These formulas still do not provide a way of updating $\sin C_n$ and $\cos C_n$. Thes provided by equation 85 .

$$\sin C_n = \sin(C_{n-1} + \Delta) = (\sin C_{n-1})\left(1 - \frac{\Delta^2}{2}\right) + \Delta \cos C_{n-1}$$

$$\cos C_n = \cos(C_{n-1} + \Delta) = (\cos C_{n-1})\left(1 - \frac{\Delta^2}{2}\right) - \Delta \sin C_{n-1}$$

So

$$\sin C_n = \left[1 - (\dot{C}_n + \dot{C}_{n-1})^2 \frac{h^2}{8}\right] \sin C_{n-1} + \left[(\dot{C}_n + \dot{C}_{n-1})\frac{h}{2}\right] \cos C_{n-1}$$

$$\cos C_n = \left[1 - (\dot{C}_n + \dot{C}_{n-1})^2 \frac{h^2}{8}\right] \cos C_{n-1} - \left[(\dot{C}_n + \dot{C}_{n-1})\frac{h}{2}\right] \sin C_{n-1}$$

where $\dot{C}_n$ and $\dot{C}_{n-1}$ are expressed in radians per second.

These formulas are used for updating the position of submarines and surface ve in all turn and dive maneuvers.

A

A. 2 SURFACE VESSEL

The following equations were used as a basis for deriving the kinematic equations motion for a surface vessel. For any given vessel, $a_1, \cdots, a_8$ are constants, and are characteristic of that vessel.

$$\dot{y} = (a_1\alpha + a_2\delta)V^2 + a_3yV \tag{8}$$

$$\dot{\alpha} = (a_4\alpha + a_5\delta)V + a_6y \tag{8}$$

$$\dot{V} = a_7(a_4\alpha + a_5\delta)^2V^2 + a_8(V_o{}^2 - V^2) \tag{9}$$

y = rate of change of the ship's heading (θ). (θ is measured clockwise from due north.)

α = side-slip angle (the angle from the true direction of motion to the direction of the ship's heading. The positive direction is clockwise.)

θ = actual course angle (measured clockwise from due north. $C = \theta - \alpha$)

δ = rudder deflection angle (left rudder is positive.)

V = ship's velocity

V_o = velocity for which the ship's engines are set. (In a turn this means velocity as the ship is entering the turn. In an acceleration or deceleration maneuver it means "ordered speed.")

The constants $a_1, \cdots, a_8$, chosen from available sources, are such that all angles will be in radians and velocity in feet per second. Notice, however, that since equ tions 88 and 89 are linear in "angle measure" any unit of angle measure can be chosen for α, y and δ as long as it is consistent for all three and provided it is converted to radi measure for use in equation 90.

A. 2. 1 Surface Vessel Turn

A. 2. 1. 1 Solution of the Differential Equations

The equations for turn rate, or $\dot{C} = y - \dot{\alpha}$, are found by solving equations 88 and 89 simultaneously. V must be assumed constant during this solution, otherwise no solu tion would be possible. In the final solution V_n (the current value of V) will be used whe ever V is called for.

α is assumed to vary linearly with time.

Consider the homogeneous equations:

$$\dot{y} = a_3yV + a_1\alpha V^2$$

$$\dot{\alpha} = a_6y + a_4\alpha V \tag{9}$$

This system of equations has Jacobean Matrix

$$\begin{Vmatrix} a_3V & a_1V^2 \\ a_6 & a_4V \end{Vmatrix}$$

The equation for the eigenvalues associated with this matrix is

$$(a_3V - x)(a_4V - x) - a_1a_6V^2 = 0$$

$$x^2 - (a_3 + a_4)xV + (a_3a_4 - a_1a_6)V^2 = 0$$

$$x = \left[(a_3 + a_4) \pm \sqrt{(a_3 + a_4)^2 - 4(a_3a_4 - a_1a_6)}\right]\frac{V}{2}$$

The two eigenvalues will be denoted by Z_1 and Z_2

$$Z_1 = \left[(a_3 + a_4) + \sqrt{(a_3 + a_4)^2 - 4(a_3a_4 - a_1a_6)}\right]\frac{V}{2}$$

$$Z_2 = \left[(a_3 + a_4) - \sqrt{(a_3 + a_4)^2 - 4(a_3a_4 - a_1a_6)}\right]\frac{V}{2}$$

The solutions to the homogeneous equations are therefore,

$$y = A_1e^{Z_1t} + A_2e^{Z_2t}$$

$$\alpha = B_1e^{Z_1t} + B_2e^{Z_2t} .$$

There are two constants too many. Substituting into equation 91 ,

$$A_1Z_1e^{Z_1t} + A_2Z_2e^{Z_2t} = (a_3VA_1 + a_1V^2B_1)e^{Z_1t} + (a_3VA_2 + a_1V^2B_2)e^{Z_2t}$$

$$B_1Z_1e^{Z_1t} + B_2Z_2e^{Z_2t} = (a_6A_1 + a_4VB_1)e^{Z_1t} + (a_6A_2 + a_4VB_2)e^{Z_2t}$$

The above equations must hold for all values of t since they result from substitut in equation 91 .

Therefore,

$$A_1Z_1 = (a_3A_1 + a_1VB_1)V$$

$$A_2Z_2 = (a_3A_2 + a_1VB_2)V$$

$$B_1 Z_1 = (a_6 A_1 + a_4 V B_1)$$

$$B_2 Z_2 = (a_6 A_2 + a_4 V B_2)$$

Thus,

$$B_1 = A_1(Z_1 - a_3 V)/a_1 V^2$$

$$B_2 = A_2(Z_2 - a_3 V)/a_1 V^2$$

$$B_1 = a_6 A_1/(Z_1 - a_4 V)$$

$$B_2 = a_6 A_2/(Z_2 - a_4 V)$$

The first two equations are equivalent to the last two:

$$\frac{Z_1 - a_3 V}{a_1 V^2} = \frac{a_6}{Z_1 - a_4 V} \qquad (9$$

$$(Z_1 - a_3 V)(Z_1 - a_4 V) - a_1 a_6 V^2 = 0$$

however, this is equation 92. The solutions to equation 91 may now be written

$$y = A_1 e^{Z_1 t} + A_2 e^{Z_2 t}$$

$$= \frac{A_1(Z_1 - a_3 V)}{a_1 V^2} e^{Z_1 t} + \frac{A_2(Z_2 - a_3 V)}{a_1 V^2} e^{Z_2 t} \qquad (9$$

Consider now the inhomogeneous equations. Let the forcing function be

$$\delta = r_1 + r_2 t$$

and let the particular solutions be

$$y = P_1 + P_2 t$$

$$\alpha = q_1 + q_2 t$$

So equations 88 and 89 become

$$P_2 = (a_1[q_1 + q_2 t] + a_2[r_1 + r_2 t])V^2 + a_3 V[P_1 + P_2 t]$$

$$q_2 = (a_4[q_1 + q_2 t] + a_5[r_1 + r_2 t])V + a_6[P_1 + P_2 t]$$

These equations must hold identically for all values of t. Therefore, the follow four equations must be true.

$$P_2 = a_1 q_1 V^2 + a_2 r_1 V^2 + a_3 P_1 V$$

$$0 = a_1 q_2 V^2 + a_2 r_2 V^2 + a_3 P_2 V$$

$$q_2 = a_4 q_1 V + a_5 r_1 V + a_6 P_1$$

$$0 = a_4 q_2 V + a_5 r_2 V + a_6 P_2$$

These four equations must now be solved for P_1, P_2, and q_2. It will not be nece sary to solve for q_1 since we are interested in $\dot{\alpha}$, not in α, and q_1 drops out of the equat for $\dot{\alpha}$, as shown below.

The equations will be solved using determinants. The denominator can be found the following manner.

$$D = \begin{vmatrix} a_3 V & -1 & a_1 V^2 & 0 \\ 0 & a_3 V & 0 & a_1 V^2 \\ a_6 & 0 & a_4 V & -1 \\ 0 & a_6 & 0 & a_4 V \end{vmatrix}$$

$$D = a_3 V[a_3 V(a_4{}^2 V^2) + a_1 V^2(-a_4 a_6 V)] + a_6[-a_1 V^2(a_3 a_4 V^2 - a_1 a_6 V^2)]$$

$$D = (a_3 a_4 - a_1 a_6)^2 V^4 \qquad (9$$

$$P_1 D = \begin{vmatrix} -a_2 r_1 V^2 & -1 & a_1 V^2 & 0 \\ -a_2 r_2 V^2 & a_3 V & 0 & a_1 V^2 \\ -a_5 r_1 V & 0 & a_4 V & -1 \\ -a_5 r_2 V & a_6 & 0 & a_4 V \end{vmatrix}$$

$$P_1D = a_1V^2[-a_3V(-a_4a_5r_1V^2 - a_5r_2V) - a_6(a_2r_2V^2 + a_1a_5r_1V^3)]$$

$$- a_4V[-a_2r_1V^2(a_3a_4V^2 - a_1a_6V^2) + (-a_2a_4r_2V^3 + a_1a_5r_2V^3)]$$

$$P_1D = r_1V^5(a_3a_4 - a_1a_6)(a_1a_5 - a_2a_4) + r_2V^4[a_1(a_3a_5 - a_2a_6) + a_4(a_1a_5 - a_2a_4)]$$

$$P_1 = r_1V\frac{(a_1a_5 - a_2a_4)}{(a_3a_4 - a_1a_6)} + r_2\frac{[a_1(a_3a_5 - a_2a_6) + a_4(a_1a_5 - a_2a_4)]}{(a_3a_4 - a_1a_6)^2}$$

The notation listed below will be used in all of the following calculations.

$$\xi = \frac{(a_1a_5 - a_2a_4)}{(a_3a_4 - a_1a_6)} \tag{97}$$

$$\eta = \frac{(a_2a_6 - a_3a_5)}{(a_3a_4 - a_1a_6)} \tag{98}$$

So P_1 can be written

$$P_1 = r_1V\xi + r_2\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} \tag{99}$$

We now calculate P_2 and q_2.

$$P_2D = \begin{vmatrix} a_3V & -a_2r_1V^2 & a_1V^2 & 0 \\ 0 & -a_2r_2V^2 & 0 & a_1V^2 \\ a_6 & -a_5r_1V & a_4V & -1 \\ 0 & -a_5r_2V & 0 & a_4V \end{vmatrix}$$

$$P_2D = a_3V[a_4V(-a_2a_4r_2V^3 + a_1a_5r_2V^3)] + a_6[-a_1V^2(-a_2a_4r_2V^3 + a_1a_5r_2V^3)]$$

$$P_2D = (a_3a_4 - a_1a_6)(a_1a_5 - a_2a_4)r_2V^5$$

$$P_2 = r_2 V \xi$$

$$q_2 D = \begin{vmatrix} a_3 V & -1 & a_1 V^2 & -a_2 r_1 V^2 \\ 0 & a_3 V & 0 & -a_2 r_2 V^2 \\ a_6 & 0 & a_4 V & -a_5 r_1 V \\ 0 & a_6 & 0 & -a_5 r_2 V \end{vmatrix}$$

$$q_2 D = a_3 V[+ a_4 V(-a_3 a_5 r_2 V^2 + a_2 a_6 r_2 V^2)] + a_6[-a_1 V^2(-a_3 a_5 r_2 V^2 + a_2 a_6 r_2 V^2)]$$

$$q_2 D = (a_3 a_4 - a_1 a_6)(a_2 a_6 - a_3 a_5) V^4 r_2$$

$$q_2 = r_2 \eta \qquad (1$$

So the equations for y and $\dot{\alpha}$ are

$$y = A_1 e^{Z_1 t} + A_2 e^{Z_2 t} + r_1 V \xi + r_2 \frac{a_4 \xi - a_1 \eta}{a_3 a_4 - a_1 a_6} + r_2 V \xi t \qquad (1$$

$$\dot{\alpha} = A_1 \frac{(Z_1 - a_3 V)}{a_1 V^2} Z_1 e^{Z_1 t} + A_2 \frac{(Z_2 - a_3 V)}{a_1 V^2} Z_2 e^{Z_2 t} + r_2 \eta \qquad (1$$

A. 2. 1. 2 The Integral of $\dot{C}$

The integral of $\dot{C} = y - \dot{\alpha}$ will be used in the last three control situations. Specific various quantities will be equated to

$\lim_{x \to \infty} \int_0^x \dot{C}\, dt$ where $\dot{C} = y - \dot{\alpha}$. The solution of this integral is not trivial. Th following two forms of equations 102 and 103 are used: (1) $r_1 = 0$ and $r_2 = \underline{\delta}/T$, wher T is the time it takes the rudder to move from $\delta = 0$ to $\delta = \underline{\delta}$, the rudder angle correspon ing to ${}_0\dot{C}$; and (2) $r_1 = \underline{\delta}$ and $r_2 = 0$. These are the two rudder conditions involved when a surface vessel goes into a turn from a $\dot{C} = 0$ condition.

Consider first the constants A_1 and A_2 for the first T seconds. $y = 0$, $r_1 = 0$ and $\alpha = 0$ when $t = 0$.

$$\text{Let } r_2 \frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_q} = P \tag{104}$$

So equation 102 becomes

$$0 = A_1 + A_2 + P$$

while equation 103 is

$$0 = A_1 Z_1 \frac{(Z_1 - a_3V)}{a_1V^2} + A_2 Z_2 \frac{(Z_2 - a_3V)}{a_1V^2} + q_2$$

$$D = \begin{vmatrix} 1 & 1 \\ Z_1 \dfrac{(Z_1 - a_3V)}{a_1V^2} & Z_2 \dfrac{(Z_2 - a_3V)}{a_1V^2} \end{vmatrix}$$

$$D = Z_2 \frac{(Z_2 - a_3V)}{a_1V^2} - Z_1 \frac{(Z_1 - a_3V)}{a_1V^2} = \frac{(Z_2 - Z_1)}{a_1V^2}(Z_2 + Z_1 - a_3V)$$

according to equation 93 , $Z_2 + Z_1 - a_3V = a_4V$. Therefore,

$$D = (Z_2 - Z_1)\frac{a_4V}{a_1V^2}$$

$$A_1D = \begin{vmatrix} -P & 1 \\ -q_2 & Z_2 \dfrac{(Z_2 - a_3V)}{a_1V^2} \end{vmatrix}$$

$$A_2D = \begin{vmatrix} 1 & -P \\ Z_1 \dfrac{(Z_1 - a_3V)}{a_1V^2} & -q_2 \end{vmatrix}$$

$$A_1 = \frac{a_1 V^2}{(Z_2 - Z_1)a_4 V}\left[q_2 - PZ_2 \frac{(Z_2 - a_3 V)}{a_1 V^2}\right]$$

$$A_2 = \frac{-a_1 V^3}{(Z_2 - Z_1)a_4 V}\left[q_2 - PZ_1 \frac{(Z_1 - a_3 V)}{a_1 V^2}\right]$$

Let A_1 and A_2 be the constants used in equations 102 and 103 while δ is cha and let A_1' and A_2' be the constants used when δ has attained its fixed value ($\underline{\delta}$). The two sets of constants because r_1 and r_2 change value, therefore, there are two differe of equations. However, when $t = T$, the two values of y and $\dot{\alpha}$ expressed by these two of equations must be equal.

A_1' and A_2' are defined so that, when $t = T$,

$$y_T = A_1' + A_2' + \underline{\delta} V \xi$$

$$\alpha_T = A_1' \; Z_1 \frac{(Z_1 - a_3 V)}{a_1 V^2} + A_2' \; Z_2 \frac{(Z_2 - a_3 V)}{a_1 V^2}$$

But A_1 and A_2 are defined so that, when $t = T$

$$y_T = A_1 e^{Z_1 T} + A_2 e^{Z_2 T} + r_2 V \xi T + P$$

$$\dot{\alpha}_T = A_1 Z_1 \frac{(Z_1 - a_3 V)}{a_1 V^2} e^{Z_1 T} + A_2 Z_2 \frac{(Z_2 - a_3 V)}{a_1 V^2} e^{Z_2 T} + q_2$$

where $\underline{\delta} = r_2 T$

The two can now be equated and A_1' and A_2' can be found.

$$D = \begin{vmatrix} 1 & 1 \\ Z_1 \frac{(Z_1 - a_3 V)}{a_1 V^2} & Z_2 \frac{(Z_2 - a_3 V)}{a_1 V^2} \end{vmatrix}$$

$$D = (Z_2 - Z_1) \frac{a_4 V}{a_1 V^2}$$

$$A_1'D = \begin{vmatrix} A_1 e^{Z_1 T} + A_2 e^{Z_2 T} + P & 1 \\ A_1 Z_1 \left(\dfrac{Z_1 - a_3 V}{a_1 V^2}\right) e^{Z_1 T} + A_2 Z_2 \left(\dfrac{Z_2 - a_3 V}{a_1 V^2}\right) e^{Z_2 T} + q_2 & Z_2 \left(\dfrac{Z_2 - a_3 V}{a_1 V^2}\right) \end{vmatrix}$$

$$A_1'D = A_1(Z_2 - Z_1)\ \frac{a_4 V}{a_1 V^2}\ e^{Z_1 T} - \left[q_2 - P Z_2 \frac{(Z_2 - a_3 V)}{a_1 V^2} \right]$$

$$A_1' = A_1(e^{Z_1 T} - 1) \tag{107}$$

Similarly,

$$A_2' = A_2(e^{Z_1 T} - 1) \tag{108}$$

Therefore,

$$\int_0^x \dot{C}\, dt = \int_0^T \left\{ A_1 \left[1 - Z_1 \left(\frac{Z_1 - a_3 V}{a_1 V^2} \right) \right] e^{Z_1 t} + A_2 \left[1 - Z_2 \left(\frac{Z_2 - a_3 V}{a_1 V^2} \right) \right] e^{Z_2 t} \right.$$

$$\left. + r_2 V \xi t + P - q_2 \right\} dt$$

$$+ \int_T^x \left\{ A_1(e^{Z_1 T} - 1) \left[1 - Z_1 \left(\frac{Z_1 - a_3 V}{a_1 V^2} \right) \right] e^{Z_1 (t-T)} + \underline{\delta} V \xi \right.$$

$$\left. + A_2(e^{Z_2 T} - 1) \left[1 - Z_2 \left(\frac{Z_2 - a_3 V}{a_1 V^2} \right) \right] e^{Z_2 (t-T)} \right\} dt$$

$$= \frac{A_1}{Z_1} \left[1 - Z_1 \left(\frac{Z_1 - a_3 V}{a_1 V^2} \right) \right] \left[e^{Z_1 T} - 1 + \left(e^{Z_1 T} - 1 \right) \left(e^{Z_1 (x-T)} - 1 \right) \right]$$

$$+\frac{A_2}{Z_2}\left[1-\frac{Z_2(Z_2-a_3V)}{a_1V^2}\right]\left[e^{Z_2T}-1+\left(e^{Z_2T}-1\right)\left(e^{Z_2(x-T)}-1\right)\right]$$

$$+ r_2V\xi\frac{T^2}{2} + (P - q_2)T + \underline{\delta}V\xi(x - T)$$

As $x \rightarrow \infty$, this has, as its limit

$$r_2V\xi\frac{T^2}{2} + (P - q_2)T + \underline{\delta}V\xi(x - T)$$

Then, since $r_2T = \underline{\delta}$

$$\lim_{x\rightarrow\infty}\int_0^x \dot{C}\,dt = -\xi V\underline{\delta}\frac{T}{2} + (P - q_2)T + \underline{\delta}V\xi x$$

This equation is somewhat misleading since it gives the impression that, as $T \rightarrow 0$, $\lim_{x\rightarrow\infty}\int_0^x \dot{C}\,dt \rightarrow \underline{\delta}V\xi x$. This is not true, because, as $T \rightarrow 0$, $r_2 \rightarrow \infty$ because $r_2 = \underline{\delta}/T$. Thus P and $q_2 \rightarrow \infty$ by equations 101 and 104 . $(P - q_2)T$ is therefore an indeterminate form and must be evaluated by using the original definitions and the fact that $r_2T = \underline{\delta}$. Thus,

$$\lim_{x\rightarrow\infty}\int_0^x \dot{C}\,dt = \underline{\delta}V\xi(x - \frac{T}{2}) + \underline{\delta}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$$

This equation will be referred to in section A.2.1.3. For application in this paragraph it must be noted that ${}_0\dot{C} = \underline{\delta}V_f\xi$ (let $t \rightarrow \infty$ in equations 102 and 103 with $r_2 = 0$ and $r_1 = \underline{\delta}$), where V_f is the final, steady-state value of V.

Since V is assumed constant in these derivations, the form of the integral which will be used is

$$\lim_{x\rightarrow\infty}\int_0^x \dot{C}\,dt = {}_0\dot{C}(x - \frac{T}{2}) + \frac{{}_0\dot{C}}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right] \qquad (109)$$

V_f will be derived in section A.2.2.2.

A.2.1.3 Four Control Situations

Operator Control

In the operator control situation, equations 102 and 103 will be solved directly to find y_n and $\dot{\alpha}_n$ in terms of y_{n-1} and $\dot{\alpha}_{n-1}$. Consider the equations for y_n, $\dot{\alpha}_n$, y_{n-1} and $\dot{\alpha}_{n-1}$. (In equation 102, $r_1 V\xi + r_2 V\xi t$ can be written $\delta V\xi$ since $r_1 + r_2 t = \delta$).

$$y_n - A_1 e^{Z_1 nh} - A_2 e^{Z_2 nh} = \delta_n V\xi + r_2\left(\frac{a_4\xi - a_1\eta}{a_3 a_4 - a_1 a_6}\right)$$

The last term in this equation appears frequently.

Let $\quad P = r_2\left(\dfrac{a_4\xi - a_1\eta}{a_3 a_4 - a_1 a_6}\right) \qquad$ (equation 104)

$$(-A_1 e^{Z_1 nh})e^{-Z_1 h} + (-A_2 e^{Z_2 nh})e^{-Z_2 h} = \delta_{n-1} V\xi + P - y_{n-1}$$

$$\dot{\alpha}_n + (-A_1 e^{Z_1 nh})Z_1\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right) + (-A_2 e^{Z_2 nh})Z_2\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) = r_2\eta$$

Let $q_2 = r_2\eta$ (equation 101)

$$(-A_1 e^{Z_1 nh})e^{-Zh} Z_1\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right) + (-A_2 e^{Z_2 nh})e^{-Z_2 h} Z_2\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) = q_2 - \dot{\alpha}_{n-1}$$

These four equations in the four unknowns $y_n, \dot{\alpha}_n, (-A_1 e^{Z_1 nh})$ and $(-A_2 e^{Z_2 nh})$ will be solved for y_n and $\dot{\alpha}_n$. The determinant for the denominator is:

$$D = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & e^{-Z_1 h} & e^{-Z_2 h} \\ 0 & 1 & Z_1\left(\dfrac{Z_1 - a_3 V}{a_1 V^2}\right) & Z_2\left(\dfrac{Z_2 - a_3 V}{a_1 V^2}\right) \\ 0 & 0 & e^{-Z_1 h} Z_1\left(\dfrac{Z_1 - a_3 V}{a_1 V^2}\right) & e^{-Z_2 h} Z_2\left(\dfrac{Z_2 - a_3 V}{a_1 V^2}\right) \end{vmatrix}$$

$$D = -\left[e^{-Z_1 h} e^{-Z_2 h}\left(Z_2\left[\frac{Z_2 - a_3 V}{a_1 V^2}\right] - Z_1\left[\frac{Z_1 - a_3 V}{a_1 V^2}\right]\right)\right]$$

$$D = -\frac{e^{-(Z_1 + Z_2)h}}{a_1 V^2}(Z_2 - Z_1)(Z_2 + Z_1 - a_3 V)$$

However, $Z_2 + Z_1 = (a_3 + a_4)V$ (equation 103)

Therefore $D = e^{-(Z_1+Z_2)h}(Z_1 - Z_2)\dfrac{a_4 V}{a_1 V^2}$ (110)

$$y_n D = \begin{vmatrix} \delta_n V\xi + P & 0 & 1 & 1 \\ \delta_{n-1}V\xi + P - y_{n-1} & 0 & e^{-Z_1 h} & e^{-Z_2 h} \\ q_2 & 1 & Z_1\left(\dfrac{Z_1 - a_3 V}{a_1 V^2}\right) & Z_2\left(\dfrac{Z_2 - a_3 V}{a_1 V^2}\right) \\ q_2 - \dot{\alpha}_{n-1} & 0 & e^{-Z_1 h} Z_1\left(\dfrac{Z_1 - a_3 V}{a_1 V^2}\right) & e^{-Z_2 h} Z_2\left(\dfrac{Z_2 - a_3 V}{a_1 V^2}\right) \end{vmatrix}$$

$$\begin{aligned} y_n D = -\Bigg[&(\delta_n V\xi + P)e^{-(Z_1+Z_2)h}(Z_2 - Z_1)\frac{a_4 V}{a_1 V^2} \\ &- (\delta_{n-1}V\xi + P - y_{n-1})\left(e^{-Z_2 h} Z_2\left[\frac{Z_2 - a_3}{a_1 V^2}\right] - e^{-Z_1 h} Z_1\left[\frac{Z_1 - a_3 V}{a_1 V^2}\right]\right) \\ &+ (q_2 - \dot{\alpha}_{n-1})(e^{-Z_2 h} - e^{-Z_1 h})\Bigg] \end{aligned} \qquad (111)$$

At this point it becomes evident that to proceed further, the exponential must be expanded. Since only the first two terms of the expansion will be used, the validity of such an approximation depends on the magnitudes of Z_1 and Z_2. It should be pointed out that $e^{Z_1 h}$ and $e^{Z_2 h}$ cannot be evaluated for each vehicle and mesh size and kept as a constant. This is because Z_1 and Z_2 are both proportional to V, and since V changes during a turn maneuver, so do Z_1 and Z_2.

From the constants available, and from the nature of the equations, it is evident that a_3 and a_4 will always be negative. Therefore, Z_2 has the larger magnitude. If the approximation is valid for Z_2, it is valid for Z_1. Note that $a_3a_4 - a_1a_6$ in equation 93 must be positive otherwise Z_1 would be positive, leading to divergent solutions. It does turn out to be positive for all vessels for which constants are available.

Consider Z_2 for three of the vessels for which constants are available. Z_2 must be such that $|Z_2h| < 1$. This leads to the following restrictions on V.

Surface Vessel I $Vh <$ 86.5 feet = 51.2 knot-seconds

Surface Vessel II $Vh <$ 71.8 feet = 42.5 knot-seconds

Surface Vessel III $Vh <$ 185 feet = 110 knot-seconds

In other words, h may have to be one second or less for certain surface vessels at high speeds, whereas for others, at slow speeds, it may be as high as 5 to 10 seconds.

Assuming that Vh satisfies the appropriate condition, linear approximations will be used for the exponentials in equation 111. The terms will be approximated separately, since they will appear again later. Note that each approximation involves expansions of e^{Z_1h} and e^{Z_2h}, but never $e^{(Z_1+Z_2)h}$ as such.

$$y_n = \frac{-a_1V^2}{(Z_1 - Z_2)a_4V}\left[(\delta_nV\xi + P)(Z_2 - Z_1)\frac{a_4V}{a_1V^2}\right.$$

$$- (\delta_{n-1}V\xi + P - y_{n-1})\left(e^{Z_1h}Z_2\left[\frac{Z_2 - a_3V}{a_1V^2}\right] - e^{Z_2h}Z_1\left[\frac{Z_1 - a_3V}{a_1V^2}\right]\right)$$

$$\left. + (q_2 - \dot{a}_{n-1})\left(e^{Z_1h} - e^{Z_2h}\right)\right]$$

$$e^{Z_1h}Z_2\left[\frac{Z_2 - a_3V}{a_1V^2}\right] - e^{Z_2h}Z_1\left[\frac{Z_1 - a_3V}{a_1V^2}\right]$$

$$= Z_2\left[\frac{Z_2 - a_3V}{a_1V^2}\right] - Z_1\left[\frac{Z_1 - a_3V}{a_1V^2}\right] + Z_1Z_2h\left[\frac{Z_2 - Z_1}{a_1V^2}\right]$$

$$= \frac{(Z_2 - Z_1)}{a_1V^2}[Z_2 + Z_1 - a_3V + Z_1Z_2h]$$

$$= \frac{(Z_2 - Z_1)}{a_1V^2}[a_4V + (a_3a_4 - a_1a_6)hV^2]$$

The last step is accomplished by substituting Z_1 and Z_2 from equation 93.

$$\left(e^{Z_1 h} - e^{Z_2 h}\right) = h(Z_1 - Z_2) \tag{112}$$

So equation 111 becomes

$$y_n = (\delta_n V\xi + P) - (\delta_{n-1}V\xi + P - y_{n-1})\left[1 + \frac{(a_3a_4 - a_1a_6)}{a_4}Vh\right] - (q_2 - \dot{\alpha}_{n-1})\frac{a_1 Vh}{a_4} \tag{113}$$

This may be simplified further. From equation 104, the P term in the above equation becomes:

$$-r_2\left(\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6}\right)\left(\frac{a_3a_4 - a_1a_6}{a_4}\right)Vh = -r_2\xi Vh + \frac{a_1r_2\eta Vh}{a_4}$$

However

$$r_2\xi Vh = (\delta_n - \delta_{n-1})\xi V \quad \text{since} \quad \delta = r_1 + r_2 t,$$

and

$$\frac{a_1r_2\eta Vh}{a_4} = \frac{a_1q_2Vh}{a_4} \quad \text{from equation 101.}$$

So equation 113 becomes:

$$y_n - y_{n-1} = \frac{Vh}{a_4}\left[(y_{n-1} - V\xi\delta_{n-1})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right] \tag{114}$$

Now consider $\dot{\alpha}_n$.

$$\dot{\alpha}_n D = \begin{vmatrix} 1 & \delta_n V\xi + P & 1 & 1 \\ 0 & \delta_{n-1}V\xi + P - y_{n-1} & e^{-Z_1 h} & e^{-Z_2 h} \\ 0 & q_2 & Z_1\left(\dfrac{Z_1 - a_3V}{a_1V^2}\right) & Z_2\left(\dfrac{Z_2 - a_3V}{a_1V^2}\right) \\ 0 & q_2 - \dot{\alpha}_{n-1} & e^{-Z_1 h}Z_1\left(\dfrac{Z_1 - a_3V}{a_1V^2}\right) & e^{-Z_2 h}Z_2\left(\dfrac{Z_2 - a_3V}{a_1V^2}\right) \end{vmatrix}$$

$$\dot{\alpha}_n D = (\delta_{n-1} V\xi + P - y_{n-1}) Z_1 Z_2 \left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right)\left(e^{-Z_2 h} - e^{-Z_1 h}\right)$$

$$-q_2 e^{-(Z_1+Z_2)h}\left[Z_2\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) - Z_1\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)\right]$$

$$+ (q_2 - \dot{\alpha}_{n-1})\left[e^{-Z_1 h} Z_2\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) - e^{-Z_2 h} Z_1\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)\right] \tag{115}$$

Two terms in equation 115 have not yet been simplified.

$$\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) = \frac{1}{a_1^2 V^4}\left[Z_1 Z_2 - a_3 V(Z_1 + Z_2) + a_3^2 V^2\right]$$

$$= \frac{V^2}{a_1^2 V^4}\left[(a_3 a_4 - a_1 a_6) - a_3(a_3 + a_4) + a_3^2\right] = \frac{-a_6}{a_1 V^2} \tag{116}$$

$$e^{(Z_1+Z_2)h}\left[e^{-Z_1 h} Z_2\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) - e^{-Z_2 h} Z_1\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)\right]$$

$$= (Z_2 - Z_1)\frac{a_4 V}{a_1 V^2} + Z_2^2 h\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) - Z_1^2 h\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)$$

$$= \frac{(Z_2 - Z_1)}{a_1 V^2}\left[a_4 V + \left((Z_2^2 + Z_1 Z_2 + Z_1^2) - a_3 V(Z_2 + Z_1)\right) h\right]$$

$$Z_2^2 + Z_1 Z_2 + Z_1^2 - a_3 V(Z_2 + Z_1)$$

$$= \left[(a_3 + a_4)^2 - 2(a_3 a_4 - a_1 a_6) + (a_3 a_4 - a_1 a_6) - a_3(a_3 + a_4)\right] V^2$$

$$= \left[a_4^2 + a_1 a_6\right] V^2$$

Therefore,

$$e^{(Z_1+Z_2)h}\left[e^{-Z_1 h} Z_2\left(\frac{Z_2 - a_3 V}{a_1 V^2}\right) - e^{-Z_2 h} Z_1\left(\frac{Z_1 - a_3 V}{a_1 V^2}\right)\right]$$

$$= \frac{(Z_2 - Z_1)}{a_1 V^2}\left(a_4 V + (a_4^2 + a_1 a_6) V^2 h\right) \tag{117}$$

Hence,

$$\dot{\alpha}_n = (\delta_{n-1}V\xi + P - y_{n-1})\left(-a_6(a_3a_4 - a_1a_6)\frac{Vh}{a_4}\right)$$

$$+ q_2 - (q_2 - \dot{\alpha}_{n-1})[1 + (a_4^{\,2} + a_1a_6)\frac{Vh}{a_4}] \tag{118}$$

when equations 110, 113, 116, and 117 are used with equation 115.

Equation 118 may be further simplified. Consider the P and q_2 terms

$$[-a_6P(a_3a_4 - a_1a_6) - q_2(a_4^{\,2} + a_1a_6)]\frac{Vh}{a_4}$$

By equations 101 and 104 this becomes

$$\frac{-Vhr_2}{a_4}[a_6(a_4\xi - a_1\eta) + \eta(a_4^{\,2} + a_1a_6)] = -Vhr_2[a_6\xi + a_4\eta]$$

However by equations 97 and 98,

$$a_6\xi + a_4\eta = \frac{a_6(a_1a_5 - a_2a_4) + a_4(a_2a_6 - a_3a_5)}{(a_3a_4 - a_1a_6)} = -a_5 \tag{119}$$

So

$$-Vhr_2[a_6\xi + a_4\eta] = a_5Vhr_2$$

and

$$\dot{\alpha}_n - \dot{\alpha}_{n-1} = \frac{Vha_6}{a_4}[(y_{n-1} - V\xi\delta_{n-1})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}] + Vh(a_4\dot{\alpha}_{n-1} + a_5r_2) \tag{120}$$

Equation 114 is repeated for the sake of comparison

$$y_n - y_{n-1} = \frac{Vh}{a_4}[(y_{n-1} - V\xi\delta_{n-1})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}]$$

and $\quad r_2 = (\delta_n - \delta_{n-1})/h \qquad (121)$

Command Control

In the command control situation the input is in the form ${}_o\dot{C}$ rather than δ. The command control equation is comprised of equations 114 and 120 rewritten with $V\xi\delta_{n-1} = {}_o\dot{C}$ and $r_2 = 0$. This corresponds to entering a turn by an instantaneous change of rudder angle from 0 to $\underline{\delta}$ at time $t = 0$. It is evident that the resultant motion will differ somewhat from the situation in which the rudder moves at the rate $\underline{\delta}/T$ for a time T. That the total change in course at some future time (x) will be different for these two rudder histories can be seen by looking at the coefficient of T in equation 109. In order to compensate for this difference, a time delay will be used in starting the maneuver.

Consider equation 109,

$$\lim_{x\to\infty}\int_0^x \dot{C}\,dt = {}_0\dot{C}(x - \tfrac{T}{2}) + \frac{{}_0\dot{C}}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$$

Letting $T = 0$ is equivalent to the physical situation where $\delta = \underline{\delta}$ at $t = 0$, and $r_2 = 0$ throughout. However, this is exactly what happens in the command control situation. Therefore, in the command control situation, values of C for large t (when $\dot{C} \approx \underline{\delta}V\xi$) are attained T/2 seconds sooner than in the operator control situation. However, a time delay of T/2 in the command control formulation will make the two values of C equal.

Thus, the command control equations are

$y_n = 0$ when $nh < T/2$

$$y_n = y_{n-1} + \frac{Vh}{a_4}\left[(y_{n-1} - {}_0\dot{C})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right] \tag{122}$$

when $nh \geq T/2$

$\dot{\alpha}_n = 0$ when $nh < T/2$

$$\dot{\alpha}_n = \dot{\alpha}_{n-1} + \frac{Vha_6}{a_4}\left[(y_{n-1} - {}_0\dot{C})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right] + Vha_4\dot{\alpha}_{n-1}$$

when $nh \geq T/2$ (123)

<u>Instructor Control</u>

In the instructor control situation, $\dot{C}$ is increased linearly until it equals ${}_0\dot{C}$. The rate of increase is such that C at some future time (x) is equal to the integral in equation 109. Let m be the rate of increase of $\dot{C}$.

$$\lim_{x\to\infty}\left\{\int_0^{{}_0\dot{C}/m} mt\,dt + \int_{{}_0\dot{C}/m}^{x} {}_0\dot{C}\,dt\right\} = \lim_{x\to\infty}\int_0^x \dot{C}\,dt \tag{124}$$

$$\left(\frac{{}_0\dot{C}}{m}\right)^2\frac{m}{2} + {}_0\dot{C}\left(x - \frac{{}_0\dot{C}}{m}\right) = {}_0\dot{C}(x - \tfrac{T}{2}) + \frac{{}_0\dot{C}}{V_f\xi}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$$

$$\frac{{}_0\dot{C}}{2m} = -\frac{T}{2} + \frac{1}{V_f\xi}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$$

$$m = {}_0\dot{C}\Big/\left[T - \frac{2}{V_f\xi}\left(\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right)\right] \tag{125}$$

Therefore, the turn rate is updated as follows:

$$\dot{C}_n = \dot{C}_{n-1} + h_o\dot{C}/\left[T - \frac{2}{V_f\xi}\left(\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right)\right]$$

as long as $|\dot{C}_n| < |{}_o\dot{C}|$

$$\dot{C}_n = {}_o\dot{C} \quad \text{otherwise.} \tag{126}$$

Program Control

In the program control situation, $\dot{C}$ goes abruptly from zero to ${}_o\dot{C}$ after an appropriate time delay. The time delay is chosen so that:

$$\lim_{x\to\infty} \int_{\tau}^{x} {}_o\dot{C}\, dt = \lim_{x\to\infty} \int_{0}^{x} \dot{C}\, dt$$

where the second integral is derived from equation 109.

Therefore,

$${}_o\dot{C}(x - \tau) = {}_o\dot{C}(x - \frac{T}{2}) + \frac{{}_o\dot{C}}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$$

$$\tau = \frac{T}{2} - \frac{1}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right] \tag{127}$$

Hence,

$$\dot{C}_n = 0 \text{ when } nh < \frac{T}{2} - \frac{1}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$$

$$\dot{C}_n = {}_o\dot{C} \text{ when } nh \geq \frac{T}{2} - \frac{1}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right] \tag{128}$$

NOTE: If T is not known it may be estimated, provided a value is known for δ or δ_{max}.

Since ${}_o\dot{C} = \underline{\delta}\xi V_f$,

$$T = {}_o\dot{C}/\xi V_f\,\dot{\delta}$$

V_f will be derived in paragraph A. 2. 2. 2.

A. 2. 2 Surface Vessel Speed

A. 2. 2. 1 Solution of the Differential Equation

Equation 90 will be solved to find the kinematic equation for surface vessel speed. In order to do this, $a_4\alpha + a_5\delta$ is assumed constant for the analytic solution, but the current values of α and δ are used at each iteration of the difference equation. Equation 90 is written:

$$\frac{dV}{a_8(V_o^2 - V^2) + a_7(a_4\alpha + a_5\delta)^2 V^2} = dt \tag{129}$$

a_7 must be negative since V is decreased when the surface vessel enters a turn. a_8 is positive since V is increased in a straight-ahead maneuver where $V_o > V$ and $\alpha = \delta = 0$ (refer to discussion after equation 90). Therefore $a_7(a_4\alpha + a_5\delta)^2 - a_8 < 0$ and equation 129 can be written

$$dV\left\{\frac{A}{\sqrt{a_8}V_o + \sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}\ V} + \frac{B}{\sqrt{a_8}V_o - \sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}\ V}\right\} = dt$$

$$A = B = \frac{1}{2V_o\sqrt{a_8}}$$

$$\frac{V_o\sqrt{a_8} + V\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}}{V_o\sqrt{a_8} - V\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}} = K_4 \exp[2V_o t\sqrt{a_8}\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}\,]$$

Letting $V = V_{n-1}$ when $t = (n-1)h$ and $V = V_n$ when $t = nh$ gives

$$\frac{V_o\sqrt{a_8} + V_n\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}}{V_o\sqrt{a_8} - V_n\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}} = \frac{V_o\sqrt{a_8} + V_{n-1}\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}}{V_o\sqrt{a_8} - V_{n-1}\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}}$$

$$\times \exp[\,2V_o h\sqrt{a_8}\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}\,] \tag{130}$$

Let $\quad a = V_o\sqrt{a_8}$ and $b = \sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}$

So

$$V_n = \frac{a}{b}\left[\frac{(a + bV_{n-1})e^{2abh} - (a - bV_{n-1})}{(a + bV_{n-1})e^{2abh} + (a - bV_{n-1})}\right]$$

$$V_n - V_{n-1} = \frac{(a^2 - V_{n-1}^2 b^2)(e^{2abh} - 1)}{b[(a + bV_{n-1})e^{2abh} + (a - bV_{n-1})]} \tag{131}$$

In order to proceed it will be necessary to approximate the exponential. The max mum value of the argument is attained when $a_4\alpha + a_5\delta$ is a maximum. This happens when has attained its steady state value in a turn, that is, when $\dot{\alpha} = 0$. Using equation 89, this happens when $(a_4\alpha + a_5\delta)V + a_6 y = 0$ and y has its steady-state value $V\xi\underline{\delta}$ (refer to discussion before equation 109). Therefore 2abh is largest when $a_4\alpha + a_5\delta = -a_6\xi\underline{\delta}$. The linear expansion will be adequate as long as $2abh < 1$. This means that

$$2V_o\sqrt{a_8}h\sqrt{a_8 - a_7 a_6^2 \xi^2 \underline{\delta}^2} < 1 \tag{132}$$

If $V_o h$ satisfies inequality 132, then the exponential in equation 131 can be expande

$$V_n - V_{n-1} = \frac{h(a^2 - V_{n-1}^2 b^2)}{1 + bh(a + bV_{n-1})} \tag{133}$$

However we wish to find an expression for $V_n - V_{n-1}$ which is linear in h, therefor equation 133 must be approximated by

$$V_n - V_{n-1} = h(a^2 - V_{n-1}^2 b^2)$$

$$V_n - V_{n-1} = h[a_8(V_o^2 - V_{n-1}^2) + a_7(a_4\alpha + a_5\delta)^2 V_{n-1}^2] \tag{134}$$

This approximation has validity if $bh(a + bV_{n-1}) < 1$. V_{n-1} is maximum when it is V_o, so the condition

$$V_o h\sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}\left(\sqrt{a_8} + \sqrt{a_8 - a_7(a_4\alpha + a_5\delta)^2}\right) < 1 \text{ must be satisfied.} \tag{135}$$

Note that this is a more stringent condition then inequality 132, therefore if satisfied, then the whole computation is valid Inequality 135 leads to the following condition fo $V_o h$ for the three surface vessels teste respectively (using $\underline{\delta}$ = 35 degrees).

Surface Vessel I $V_o h < 938.1$ feet = 555.4 knot-seconds
Surface Vessel II $V_o h < 781.3$ feet = 462.6 knot-seconds
Surface Vessel III $V_o h < 983.9$ feet = 582.5 knot-seconds (136)

If these inequalities hold, then equation 134 will be valid for acceleration/deceleration as well as for turning maneuvers. In an acceleration maneuver, V_o represents the final speed (speed for which the engines are set). In a deceleration maneuver however, th inequalities must hold for the initial speed rather than for V_o.

A.2.2.2 Four Control Situations

Operator and Command Control

Equation 134 will be used, with one slight modification, for both the operator contr and command control situations. Since α will not be one of the outputs (refer to equations

120 through 123) $a_4\alpha + a_5\delta$ will be expressed in terms of $\dot{\alpha}$ and y using equation 89. Equation 134 becomes

$$V_n - V_{n-1} = h[a_8(V_o^2 - V_{n-1}^2) + a_7(\dot{\alpha}_{n-1} - a_6 y_{n-1})^2] \tag{137}$$

The Integral of V

Development of the instructor control and program control situations requires the evaluation of $\lim_{x\to\infty} \int_0^x V\,dt$. To do this an expression must be found for V which is integrable. A constant value must be assumed for $(a_4\alpha + a_5\delta)V$. This value will be found by using $\dot{\alpha} - a_6 y = (a_4\alpha + a_5\delta)V$ (equation 89). The steady-state value in a turn will be used for $\dot{\alpha}_{n-1} - a_6 y_{n-1}$. This is $-a_{6\,o}\dot{C}$, since $\dot{\alpha} \to 0$ and $\dot{C} = y - \dot{\alpha}$.

Therefore equation 90 becomes

$$\dot{V} = a_7 a_6^2\,{}_o\dot{C}^2 + a_8(V_o^2 - V^2) \tag{138}$$

Note that, when $\dot{V} = 0$, $a_8 V = \sqrt{a_8 V_o^2 + a_6^2\,{}_o\dot{C}^2 a_7}$.

This is the steady-state value of V in a turn. It will be denoted by V_f,

$$V_f = \sqrt{V_o^2 + a_6^2\,{}_o\dot{C}^2 a_7/a_8} \tag{139}$$

Occasionally it will be necessary to find V_f as a function of $\underline{\delta}$ rather than ${}_o\dot{C}$. Since ${}_o\dot{C} = V_f \xi \underline{\delta}$,

$$V_f^2 = V_o^2 + a_6^2 V_f^2 \xi^2 \underline{\delta}^2 a_7/a_8$$

$$V_f^2 = V_o^2 \Big/ \left(1 - \frac{a_7}{a_8} a_6^2 \xi^2 \underline{\delta}^2\right)$$

$$V_f = V_o \Big/ \sqrt{1 - \frac{a_7}{a_8} a_6^2 \xi^2 \underline{\delta}^2} \tag{140}$$

Thus, equation 138 may be written

$$\dot{V} = a_8(V_f^2 - V^2) \tag{141}$$

Note that for an acceleration/deceleration maneuver, this is always the case, not an approximation.

$$dV\left[\frac{A}{V_f - V} + \frac{B}{V_f + V}\right] = a_8\,dt$$

$$A = B = \frac{1}{2V_f}$$

$$\frac{V_f + V}{V_f - V} = K_5 e^{2a_8 V_f t}$$

V will start from V_o.

$$\frac{V_f + V}{V_f - V} = \left(\frac{V_f + V_o}{V_f - V_o}\right) e^{2a_8 V_f t}$$

$$V = V_f \left[\frac{(V_f + V_o)e^{2a_8 V_f t} - (V_f - V_o)}{(V_f + V_o)e^{2a_8 V_f t} + (V_f - V_o)}\right] \tag{142}$$

$$V = V_f \left[\frac{(V_f + V_o)e^{2a_8 V_f t} + (V_f - V_o) - 2(V_f - V_o)}{(V_f + V_o)e^{2a_8 V_f t} + (V_f - V_o)}\right]$$

$$\lim_{x \to \infty} \int_0^x V\, dt = V_f x - 2(V_f - V_o)V_f \left\{\lim_{x \to \infty} \int_0^x \frac{dt}{(V_f + V_o)e^{2a_8 V_f t} + (V_f - V_o)}\right\}$$

where

$$\lim_{x \to \infty} \int_0^x \frac{dt}{(V_f + V_o)e^{2a_8 V_f t} + (V_f - V_o)}$$

$$= \lim_{x \to \infty} \frac{1}{2a_8 V_f (V_f - V_o)} \left[2a_8 V_f t - \log\left(\frac{(V_f + V_o)e^{2a_8 V_f t} + (V_f - V_o)}{2V_f}\right)\right]_0^x$$

$$= \lim_{x \to \infty} \frac{1}{2a_8 V_f (V_f - V_o)} \left[2a_8 V_f x - \log\left(\frac{(V_f + V_o)e^{2a_8 V_f x} + (V_f - V_o)}{2V_f}\right)\right]$$

Now,

$$\lim_{x \to \infty} \left[2a_8 V_f x - \log\left(\frac{(V_f + V_o)e^{2a_8 V_f x} + (V_f - V_o)}{2V_f}\right)\right]$$

$$= \lim_{x\to\infty}\left[2a_8V_fx - \log\left(\frac{(V_f + V_o)e^{2a_8V_fx}}{2V_f}\right)\right]$$

$$= \lim_{x\to\infty}\left[2a_8V_fx - 2a_8V_fx - \log\left(\frac{V_f + V_o}{2V_f}\right)\right]$$

$$= -\log\left(\frac{V_f + V_o}{2V_f}\right)$$

Therefore,

$$\lim_{x\to\infty}\int_0^x V\,dt = V_fx + \frac{2(V_f - V_o)V_f}{2a_8V_f(V_f - V_o)}\log\left(\frac{V_f + V_o}{2V_f}\right) = V_fx + \frac{1}{a_8}\log\left(\frac{V_f + V_o}{2V_f}\right)$$

$$\lim_{x\to\infty}\int_0^x V\,dt = V_fx - \frac{1}{a_8}\log\left(1 + \frac{V_f - V_o}{V_f + V_o}\right) \tag{143}$$

$V_f - V_o$ will always be less than $V_f + V_o$ since the speeds we are considering will always be positive. Therefore, the logarithm can be expanded. Equation 143 becomes

$$\lim_{x\to\infty}\int_0^x V\,dt = V_fx - \frac{1}{a_8}\left[\frac{V_f - V_o}{V_f + V_o} - \frac{(V_f - V_o)^2}{2(V_f + V_o)^2}\right]$$

$$\lim_{x\to\infty}\int_0^x V\,dt = V_fx + \frac{1}{a_8}(V_o - V_f)\left(\frac{V_f + 3V_o}{2(V_f + V_o)^2}\right) \tag{144}$$

This equation will be used in the derivation of the instructor control situation and program control situation models.

Instructor Control

In the instructor control situation, velocity changes from V_o, the original velocity, to V_f, the final velocity, as a linear function of time. This is accomplished in such a way that the displacement at some future time (x) is equal to what it would have been using the expression for V in equation 142.

$$\int_0^{(V_f-V_o)/m} (V_o + mt)dt + \lim_{x\to\infty} \int_{(V_f-V_o)/m}^{x} V_f\, dt = \lim_{x\to\infty} \int_0^x V\, dt \tag{145}$$

By using equation 144,

$$V_o \frac{(V_f - V_o)}{m} + \left(\frac{V_f - V_o}{m}\right)^2 \frac{m}{2} - V_f \frac{(V_f - V_o)}{m} + V_f x = V_f x + \frac{V_o - V_f}{a_8}\left[\frac{V_f + 3V_o}{2(V_f + V_o)^2}\right]$$

$$-\frac{(V_f - V_o)^2}{2m} + V_f x = V_f x + \frac{V_o - V_f}{a_8}\left[\frac{V_f + 3V_o}{2(V_f + V_o)^2}\right]$$

$$m = \frac{a_8}{2}(V_f - V_o) \Big/ \frac{V_f + 3V_o}{2(V_f + V_o)^2} \tag{146}$$

Note that this equation holds for acceleration, deceleration and turn maneuvers, where V_o = original V and V_f = final V in all cases. This is because it was derived using equations 141 and 145, for which these statements all hold.

Thus, the instructor control formula is:

$$V_n = V_{n-1} + \frac{a_8 h}{2} (V_f - V_o) \Big/ \left[\frac{V_f + 3V_o}{2(V_f+V_o)^2}\right] \quad \text{when } (V_f - V_o)(V_f - V_n) > 0$$

$$V_n = V_f \text{ otherwise.} \tag{147}$$

Program Control

In the program control situation, velocity changes abruptly from the original velocity V_o, to the final velocity, V_f. The abrupt change is timed in such a way that t' displacement at some future time (x) will be what it would have been using the expression for V in equation 14ⁿ

$$\int_0^{\tau} V_o\, dt + \lim_{x\to\infty} \int_{\tau}^{x} V_f\, dt = \lim_{x\to\infty} \int_0^x V\, dt \tag{148}$$

By using equation 144

$$(V_o - V_f)\tau + V_f x = V_f x + \frac{V_o - V_f}{a_8}\left(\frac{V_f + 3V_o}{2(V_f + V_o)^2}\right)$$

$$\tau = \frac{1}{a_8}\left(\frac{V_f + 3V_o}{2(V_f + V_o)^2}\right) \tag{149}$$

So

$$V_n = V_o \qquad \text{when } nh \leq \frac{1}{2a_8}\left(\frac{V_f + 3V_o}{(V_f + V_o)^2}\right)$$

$$V_n = V_f \qquad \text{when } nh > \frac{1}{2a_8}\left(\frac{V_f + 3V_o}{(V_f + V_o)^2}\right) \tag{150}$$

A. 3 AIRCRAFT

Equations have been developed for the three basic aircraft maneuvers: Acceleration; Coordinated turn and Climb. The methods of developing the equations will differ for each.

The following symbols are used.

a = Horizontal asymptote of the hyperbola (used to approximate the drag versus velocity curve, and hence the minimum drag, refer to Figure A-3.)

b = Vertical asymptote of the hyperbola (maximum airspeed, refer to Figure A-3)

c = Third parameter of the hyperbola (refer to equation 150)

D = Drag force on the aircraft (drag is a function of velocity)

m = Mass of the aircraft

T = Thrust supplied by the aircraft engines (assumed here to be independent of aircraft velocity)

v = Velocity (aircraft airspeed)

ϕ = Bank or roll angle of the aircraft

ω = Aircraft turn rate

$(\Delta C)_{\ell}$ = Course change while aircraft is rolling out of a coordinated turn

p = Roll rate

v_m = Level-flight airspeed corresponding to current airspeed and climb angle of the aircraft

θ = Attitude (pitch angle of the aircraft)

γ = Direction of flight in the vertical plane (Climb rate = $v \sin \gamma$)

α = Angle of attack ($\alpha = \theta - \gamma$)

C_L = Coefficient of Lift

W = Aircraft Weight

ρ = Density of the supporting medium

S = Effective lifting surface area

$k = 2W/\rho SQ$

A, B = Constants used in the airspeed versus attitude equations

$Q \approx C_L/\alpha$

A. 3. 1 Speed

The equations for speed buildup of an aircraft were derived from available thrust and required thrust versus true air speed curves for the F-100A which appeared in the UDOFT Test Report (refer to paragraph 3. 3. 3. 1). These curves had to be reduced to simple functions that could be easily integrated. At low altitudes when air speeds were greater than 250 knots, thrust available was nearly constant and thrust required versus true air speed resembled a hyperbola when graphed.

These two approximating functions were used for all four altitudes for which curves were available. These altitudes were: sea level, 15,000 feet, 25,000 feet and 35,000 feet. For higher altitudes, thrust would only remain constant for lower values of thrust with lower top speeds. Also, as the true air speed increased, the graph of the thrust required lost its hyperbolic shape. There was a tendency for the top speed at large thrust availabilities to become disproportionately larger at high altitudes.

Since the derivation ignores that occurrence, ordered speeds cannot be very high when the formulas are used at high altitudes. The upper limits will be pointed out hereafter.

Let D denote drag or thrust required, while T denotes thrust available and v is velocity. Drag versus velocity as mentioned above, is approximated by a hyperbola with asymptotes parallel to the coordinate axes:

$$(D - a)(v - b) = c \tag{151}$$

where a, b, and c are determined by fitting equation 151 to three points selected on that part of the graph nearest to a hyperbola in shape. As long as T exceeds D, the aircraft will accelerate. The speed will be constant when T = D.

Drag versus velocity hyperbola:

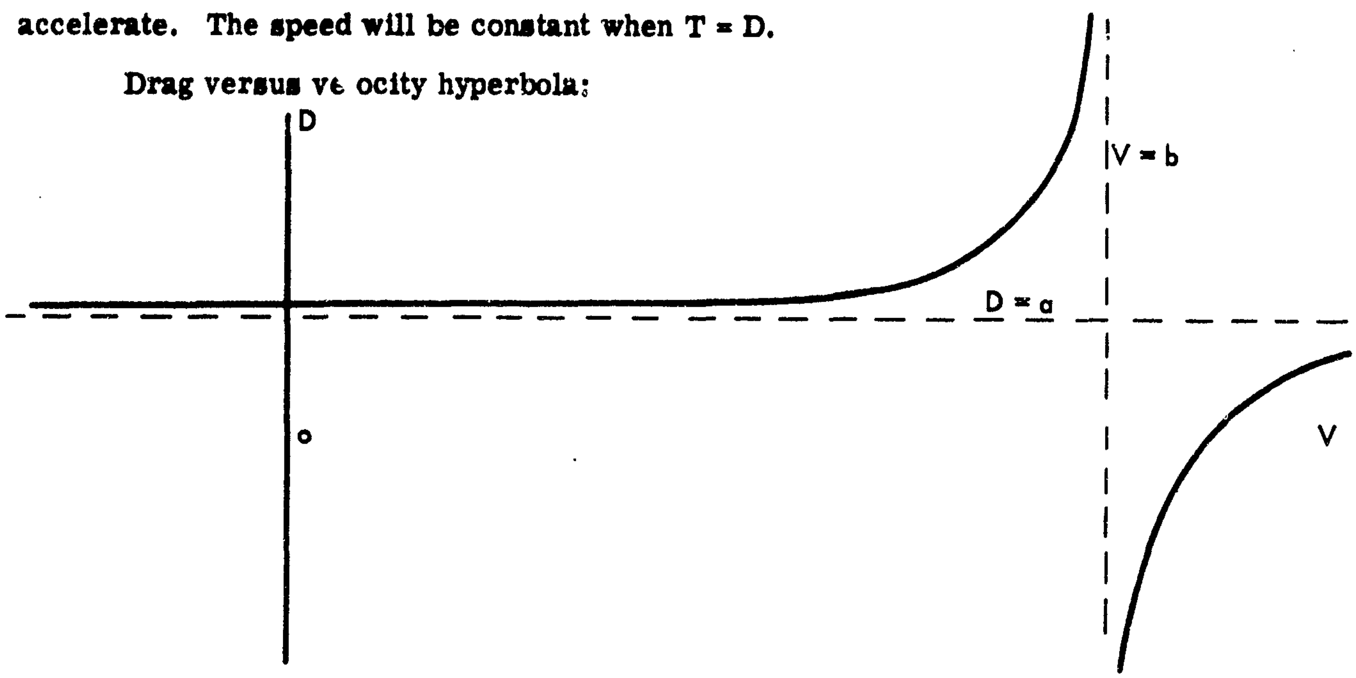

Figure A-3

If the thrust were adjusted to attain a certain ordered velocity, the differential equation would be:

$$\text{Force} = m\dot{v} = T - D \tag{152}$$

$$m\frac{dv}{dt} = D({}_ov) - D(v)$$

$$m\frac{dv}{dt} = \left(\frac{c}{{}_ov - b} + a\right) - \left(\frac{c}{v - b} + a\right)$$

$$m\frac{dv}{dt} = c\left[\frac{1}{{}_ov - b} - \frac{1}{v - b}\right] \tag{153}$$

$$dv\left[1 + \frac{{}_ov - b}{v - {}_ov}\right] = \frac{c\,dt}{m({}_ov - b)}$$

$$v + ({}_ov - b)\log({}_ov - v) + K = \frac{ct}{m({}_ov - b)} \tag{154}$$

Let $v = v_n$ when $t = t_n$ and $v = v_{n-1}$ when $t = t_{n-1}$, where $t_n - t_{n-1} = h$ for all n.

$$v_n - v_{n-1} + ({}_ov - b)\log\left[1 + \frac{v_{n-1} - v_n}{{}_ov - v_{n-1}}\right] = \frac{ch}{m({}_ov - b)} \tag{155}$$

where $1 + \dfrac{v_{n-1} - v_n}{{}_ov - v_{n-1}} = \dfrac{{}_ov - v_n}{{}_ov - v_{n-1}}$

$\dfrac{v_{n-1} - v_n}{{}_ov - v_{n-1}}$ is always less than unity because v_n is always between v_{n-1} and ${}_ov$, usually bei much closer to the former. Therefore,

$\log\left[1 + \dfrac{v_{n-1} - v_n}{{}_ov - v_{n-1}}\right]$ can be expanded in powers of $\dfrac{v_{n-1} - v_n}{{}_ov - v_{n-1}}$.

Keeping the first power only yields:

$$v_n - v_{n-1} + ({}_ov - b)\frac{v_{n-1} - v_n}{{}_ov - v_{n-1}} = \frac{ch}{m({}_ov - b)}$$

$$v_n = v_{n-1} + \frac{ch({}_ov - v_{n-1})}{m({}_ov - b)(b - v_{n-1})} \tag{156}$$

From the general shape of the hyperbola it can be seen that $v < b$ and $D < a$. Therefore c will always be negative (refer to equation 151).

If all speeds are kept below b, no serious inaccuracies will occur. If v exceeds b will be on the other branch of the hyperbola, which in no way resembles the actual drag

versus velocity curve. In fact, if v - b is a small positive number, drag will have a large negative value. This situation is impossible.

Fitting the hyperbola to the four available graphs produces the following values for b and c:

Altitude	b	c
Sea Level	711.636 knots	−885,980 knots x pounds thrust
15,000 feet	667.906 knots	−253,667 knots x pounds thrust
25,000 feet	620.070 knots	−112,651 knots x pounds thrust
35,000 feet	572.293 knots	− 22,087 knots x pounds thrust

These values must be expressed as a function of altitude. The best fit was achieved with the formulas:

$$b = 711.636\,(1 - 0.00551 h_o) \tag{157}$$

$$c = 26 h_o - 1000 \tag{158}$$

where h_o is the altitude in 1000-foot units.

A.3.2 Turn

Equations were developed for the coordinated turn of an aircraft. These equations are almost entirely independent of the type of aircraft; only the air speed and roll rate would be affected by this consideration.

The basic condition for the coordinated turn is that the four elements, bank angle (ϕ), turn rate (ω), air speed (v), and acceleration due to gravity (g) are related in the following manner:

$$\frac{\omega v}{g} = \tan \phi \tag{159}$$

This equation will hold throughout the turn.

The aircraft will start with a roll on bank angle $\phi = 0$ and then roll to ${}_o\phi$. While the aircraft is rolling into the turn $\phi = pt$ where p is the roll rate; while the aircraft is rolling out of the turn, $\phi = {}_o\phi - p(t - T)$. ${}_o\phi$, the ordered bank angle, is either directly ordered or calculated from an ordered ω by Equation 159.

Only one further calculation will be necessary. If a specific course change is desired, then the aircraft must start rolling out of the turn ahead of time. $(\Delta C)_f$ denotes the course change while the aircraft is rolling out of the turn.

$$(\Delta C)_f = \frac{g}{v} \int_0^{\frac{{}_o\phi}{p}} \tan({}_o\phi - pt)dt \tag{160}$$

$$(\Delta C)_f = \frac{g}{2v} \log(1 + \tan^2 {}_o\phi)$$

A.3.3 Climb

The derivation of equations describing the characteristics of an aircraft climb involves more approximations than that of any of the other equations thus far described. Although the equations themselves are simple, there are three separate equations and therefore three different straight line approximations of curves that are not straight lines.

The most inaccurate approximation is of the curve of attitude of pitch angle versus air speed for a given thrust. When thrust and attitude are low and air speed is greater than 250 knots, the straight line approximation will be satisfactory; otherwise it becomes somewhat inaccurate.

$$v = v_m - A(\sin\theta - B) \tag{161}$$

where v equals air speed, θ equals pitch angle and v_m equals the air speed of the aircraft flying with a zero rate of climb. At zero rate of climb, the pitch angle will not be zero; there will always be a slight difference between the pitch angle and the actual direction of flight in the vertical plane. This difference represents the angle of attack (α). The direction of flight is denoted by γ.

$$\gamma = \theta - \alpha \tag{162}$$

The lift of the aircraft is due to α. Over the range of α that is likely to be encountered here the coefficient of lift C_L will be proportional to α.

$$C_L = Q\alpha \tag{163}$$

Q will vary somewhat with Mach number, but for Mach numbers less than 0.8 remains relatively constant.

Equation 164 relates v, γ and C_L. This equation has been taken directly from Reference 4.

$$v = \sqrt{\frac{2(\cos\gamma)W/S}{\rho C_L}} \tag{164}$$

where W equals the weight of the aircraft and S its effective lifting surface area.

$\cos \gamma$ can be written $\cos(\theta - \alpha)$

$\cos(\theta - \alpha) = \cos \theta \cos \alpha + \sin \theta \sin \alpha$

since α is rarely greater than two or three degrees

$$\cos \gamma = \cos \theta + \alpha \sin \theta \tag{165}$$

Using equations 159 and 160, equation 164 can be rewritten:

$$\alpha = \frac{(2W/\rho SQ)\cos \theta}{v^2 - (2W/\rho SQ)\sin \theta} \tag{166}$$

Let $2W/\rho SQ = k$. Since ρ depends on altitude, so does k. k was evaluated by comparing θ, on the attitude versus air-speed curves already mentioned, with γ on corresponding curves of climb rate versus air speed.

Corresponding to equation 165,

$$\sin \gamma = \sin \theta - \alpha \cos \theta \tag{167}$$

so

$$\sin \gamma = \frac{v^2 \sin \theta - k}{v^2 - k \sin \theta} \tag{168}$$

and in equation 161, B equals k/v_m^2.

Equation 168 can also be written as:

$$\sin \theta = \frac{v^2 \sin \gamma + k}{v^2 + k \sin \gamma} \tag{169}$$

The formulas to be used in describing the climb maneuvers of an aircraft then, are

$$v = v_m - A(\sin \theta - \frac{k}{v_m^2}) \tag{170}$$

and

$$\sin \gamma = \frac{v^2 \sin \theta - k}{v^2 - k \sin \theta} \tag{171}$$

Since k is small compared to v^2, a second set of equations can be used for a less rigorous description.

$$v = v_m - A \sin \gamma \tag{172}$$

$$\sin \gamma = \sin \theta - \frac{k}{v^2} \tag{173}$$

There still remain several questions as to the use of these equations. For instance, given v_m, what should θ be for a desired climb rate or for an optimum climb rate?

Equations 172 and 173 produce an optimum climb rate when $\sin\gamma = \frac{v_m}{2A}$, $v = \frac{v_m}{2}$. This is obtained by finding the maximum value of $v \sin\gamma = (v_m - A \sin\gamma) \sin\gamma$ as a function of $\sin\gamma$. These values are extremely rough, but an analytic solution of equations 170 and 171 is prohibitively complex. The more complete equations produce a very complicated expression whereas the others lead to the relation:

$$\sin\gamma = \frac{v_m - \sqrt{v_m^2 - 4Av_c}}{2A} \tag{174}$$

where v_c is the maximum value of v for a given v_m.

APPENDIX B

FORMULAS

The formulas in Appendix A are here listed again, for reference. A listing to show the source of each formula is included at the end of each section. Most of the formulas are used to determine the variable at time nh (e.g., y_n) in terms of the value of the variable at time (n - 1)h (e.g., y_{n-1}). These formulas are initialized by setting the first value of y_{n-1}, for example, equal to the value of y when t = 0.

B.1 SUBMARINE

B.1.1 Symbols and Units

Symbol	Definition	Units	Positive Direction
S	speed	yards/second	forward
C	course angle	degrees	to the right of forward
D	dive angle	degrees	up from the horizontal
δ_r	rudder angle	degrees	trailing edge to the left
δ_s	stern plane angle	degrees	trailing edge down
A_1	(constant parameters of submarine's handling characteristics)	1/yards	
A_2		1/degrees	
A_3		none	
A_4		1/yards	
A_5		1/degrees	
A_6		$1/\text{yards}^2$	
A_7		1/yards	
A_8		1/degrees	
A_9		$1/\text{seconds}^2$	
A_{11}		$1/\text{yards}^2$	
X	displacement	yards	east
Y	displacement	yards	north
Z	displacement	yards	down
$_oS$	ordered speed (the speed for which the submarine's engines are set)		

Symbol	Definition
${}_{o}\dot{C}$	ordered course rate of change or steady-state rate of change of course in a turn
$\underline{\delta_r}$	rudder angle corresponding to ${}_{o}\dot{C}$
T	time required for rudder to move from zero to $\underline{\delta_r}$
S_f	steady-state speed in a turn
${}_{o}\dot{D}$	ordered dive rate
${}_{o}D$	ordered dive angle

B.1.2 Kinematic Equations of a Submarine's Motion

B.1.2.1 Acceleration/Deceleration

Instructions

${}_{o}S$ is ordered S, the final speed. S_o is the initial speed at the start of the maneuver. When $n = 1$, $S_{n-1} = S_o$.

Formulas

a. Cases 1 and 2

$$S_n - S_{n-1} = A_1 h({}_{o}S - S_{n-1})[{}_{o}S + (1 + A_3)S_{n-1}] \qquad (1)$$

b. Case 3

$$S_n - S_{n-1} = \frac{A_1 h}{2}({}_{o}S - S_o)[{}_{o}S + (1 + A_3)S_o] \quad \text{when } (S_n - {}_{o}S)(S_o - {}_{o}S) > 0$$

$$S_n = {}_{o}S \quad \text{otherwise} \qquad (2)$$

c. Case 4

$$S_n = S_o \quad \text{when } nh < 1/A_1[{}_{o}S + (1 + A_3)S_o]$$

$$S_n = {}_{o}S \quad \text{when } nh \geq 1/A_1[{}_{o}S + (1 + A_3)S_o] \qquad (3)$$

Restrictions

These formulas can only be used with positive ${}_{o}S$ and S_o. The following restriction is also imposed:

$$Sh < 1/A_1(2 + A_3) \qquad (4)$$

where S is the largest speed value that will be encountered. For the three submarines for which constants are available, the restriction becomes:

$$Sh < 250 \text{ yards} = 444 \text{ knot-seconds} \tag{5}$$

B.1.2.2 Turn

Instructions

All turn formulas give $\dot{C}$. C_n is found by using the equation:

$$C_n = C_{n-1} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1}) \tag{6}$$

In case 1 the input is speed and rudder angle. Speed input consists of ${}_oS$, the speed for which the engine is set, and S_o (S_{n-1} with n = 1). $S_o = {}_oS$ unless the turn maneuver is initiated while a previous turn maneuver or acceleration maneuver is still incomplete. Rudder angle input is $\underline{\delta_r}$, the principal rudder angle to be used in the turn, and δ_{r_n} the instantaneous value of the rudder angle. δ_{r_n} is the only input still necessary after initialization of the maneuver.

In cases 2, 3, and 4 there is a choice allowed in the form of the input. The turn may be ordered in terms of the principal rudder angle, $\underline{\delta_r}$, or in terms of an ordered turn rate, ${}_o\dot{C}$. If $\underline{\delta_r}$ is given, then equations 8 and 9 are used to find ${}_o\dot{C}$ and S_f. If ${}_o\dot{C}$ is given, then it is used directly and the following formula is used to find S_f:

$$S_f = \frac{1}{2}\left\{ {}_oS - \frac{A_2A_4|{}_o\dot{C}|}{A_6} + \sqrt{\left({}_oS - \frac{A_2A_4|{}_o\dot{C}|}{A_6}\right)^2 - \frac{4A_2A_5|{}_o\dot{C}|}{A_6}} \right\} \tag{7}$$

In the following formulas, ${}_oS$ is always the speed for which the engines are set and S_f is given by either equation 7 or equation 9. S_o is the speed entering the maneuver. It differs from ${}_oS$ only when the submarine changes from one turn rate to another, or accelerates and turns at the same time (e.g., level-up, where S_o = previous S_f and $S_f = {}_oS$).

T is the time required for the rudder to move from its pre-turn position to its position during the turn.

Formulas

a. Case

$${}_o\dot{C} = \frac{-\underline{\delta_r}}{|\underline{\delta_r}|}\, {}_oS\, \frac{\sqrt{A_4^2 + 4A_5A_6|\underline{\delta_r}|} - A_4}{2A_5(1 + A_2|\underline{\delta_r}|)} \tag{8}$$

$$S_f = {}_oS/(1 + A_2|\underline{\delta_r}|) \tag{9}$$

$$S_n - S_{n-1} = A_1 h[{}_oS - (1 + A_2|\delta_r|)S_{n-1}][{}_oS + (1 + A_2|\delta_r| + A_3)S_{n-1}] \tag{10}$$

$$\dot{C}_n - \dot{C}_{n-1} = \frac{hA_6 S_{n-1}^2 \underline{\delta_r}}{{}_o\dot{C}} \left(\frac{S_f}{S_{n-1}} \dot{C}_{n-1} - \frac{\delta_{r_{n-1}}}{\underline{\delta_r}} \, {}_o\dot{C} \right) \tag{11}$$

b. **Case 2**

$S_n = S_o$ when $nh < T/2$

$$S_n - S_{n-1} = A_1 h \frac{{}_oS}{S_f} (S_f - S_{n-1})[{}_oS + (\frac{{}_oS}{S_f} + A_3)S_{n-1}]$$

when $nh \geq T/2$ (12)

$\dot{C}_n = 0$ when $nh < T/2$

$$\dot{C}_n - \dot{C}_{n-1} = -h\frac{S_{n-1}}{S_f^2} (S_f\dot{C}_{n-1} - {}_o\dot{C}S_{n-1})(A_4 S_f + A_5|{}_o\dot{C}|)$$

when $nh \geq T/2$ (13)

c. **Case 3**

$S_n = S_o$ when $nh < T/2$

$$S_n - S_{n-1} = \frac{hA_1}{2} (S_f - S_o) \frac{{}_oS^2}{S_f} \left(\frac{{}_oS}{S_f} + A_3 + 1 \right)$$

when $nh \geq T/2$ and $(S_o - S_f)S_n > (S_o - S_f)S_f$

$S_n = S_f$ otherwise (14)

$\dot{C}_n = 0$ when $nh < T/2$

$$\dot{C}_n - \dot{C}_{n-1} = \frac{h({}_o\dot{C} - \dot{C}_o)}{2} \left[\frac{{}_oS}{S_f(A_4 S_f + A_5|{}_o\dot{C}|)} - \frac{S_o - S_f}{{}_oS^2 A_1 \left(\frac{{}_oS}{S_f} + A_3 + 1 \right)} - \frac{(S_o - S_f)(A_4 S_f + A_5|{}_o\dot{C}|)}{2A_1^2 \, {}_oS^3 \left(\frac{{}_oS}{S_f} + A_3 + 1 \right)^2} \right]^{-1}$$

when $nh \geq T/2$ and $|\dot{C}_n| < |{}_o\dot{C}|$

$\dot{C}_n = {}_o\dot{C}$ otherwise. (15)

d. Case 4

$$S_n = S_o \text{ when } nh < T/2 + S_f/{}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right)$$

$$S_n = S_f \text{ when } nh \geq T/2 + S_f/{}_oS^2A_1\left(\frac{{}_oS}{S_f} + A_3 + 1\right) \qquad (16)$$

$$\dot{C}_n = 0 \text{ when } nh < T/2 + \tau$$

$$\dot{C}_n = {}_o\dot{C} \text{ when } nh \geq T/2 + \tau \qquad (17)$$

where

$$\tau = \left[{}_oS/S_f(A_4S_f + A_5|{}_o\dot{C}|) - (S_o - S_f)/{}_oS^2A_1(\frac{{}_oS}{S_f} + A_3 + 1)\right.$$

$$\left. - (S_o - S_f)(A_4S_f + A_5|{}_o\dot{C}|)/2{}_oS^3A_1{}^2(\frac{{}_oS}{S_f} + A_3 + 1)^2\right] \qquad (18)$$

<u>Restrictions</u>

For the speed formulas to be valid, the following inequality must be valid:

$$0 < SA_1h(1 + A_2|\delta_r|)(2 + A_3 + A_2|\delta_r|) < 1 \qquad (19)$$

where S is the maximum speed encountered in the maneuver. As concerns the three submarines for which constants are available, this inequality will be satisfied if:

For Submarine I , Sh < 72.0 yards = 128 knot-seconds
For Submarine II , Sh < 93.2 yards = 166 knot-seconds
For Submarine III, Sh < 86.4 yards = 153 knot-seconds (20)

This uses $\delta_r = 35°$, the maximum attainable value of δ_r.

The corresponding inequality for the turn rate equations is:

$$(A_6\underline{\delta_r}Sh/F_c) < 1. \qquad (21)$$

where $F_c = {}_o\dot{C}/S_f$.

For Submarine I, this requires that:

Sh < 64.4 yards = 114 knot-seconds.

For the other two submarines, satisfaction of inequality 19 is sufficient.

B.1.2.3 Dive

Instructions

Speed is assumed constant in all four cases.

Case 1 requires two past values for initialization. When starting the simulation, both past values are set to zero unless the submarine is known to have a non-zero dive angle. The input at each iteration is $\delta_{s_{n-1}}$, the stern plane angle at the previous iteration.

Cases 2 and 3 produce dive angle rate as output. Dive angle is found using the formula:

$$D_n = D_{n-1} + \frac{h}{2}(\dot{D}_n + \dot{D}_{n-1}) \tag{22}$$

Case 2 can be used with either ordered dive angle rate $({}_0\dot{D})$ or current stern plane angle as input. In the former situation, the ordered dive angle is part of the initialization input; in the latter, the stern plane angle is updated at each iteration. Use of the ${}_0D$ term in the formula using the stern plane angle is optional. It is applied most effectively when ${}_0D$ is set equal to the average value of D during the maneuver. A crude estimate of this average is sufficient.

Case 3 requires an ordered dive angle $({}_0D)$ as well as an ordered dive angle rate $({}_0\dot{D})$ as initialization inputs. An alternative form is to maintain ${}_0\dot{D}$ for a prearranged time interval rather than to test D_n against ${}_0D$ at each iteration.

Case 4 requires ordered dive angle and dive angle rate, as well as initial dive angle and dive angle rate, as inputs.

Formulas

a. Case 1

$$D_n - D_{n-1} = (D_{n-1} - D_{n-2})(1 - 0.625A_7Sh) - h^2(A_9D_{n-1} + A_{11}S^2\delta_{s_{n-1}}) \tag{23}$$

b. Case 2

$$\dot{D}_n - \dot{D}_{n-1} = ({}_0\dot{D} - \dot{D}_{n-1})0.625A_7Sh \tag{24}$$

or

$$\dot{D}_n - \dot{D}_{n-1} = -h(A_{9\,0}D + A_{11}S^2\delta_{s_n} + 0.625A_7S\dot{D}_{n-1}) \tag{25}$$

$$D_n = D_{n-1} + \frac{h}{2}(\dot{D}_n + \dot{D}_{n-1})$$

c. Case 3

$$\dot{D}_n = \dot{D}_0 \text{ when } nh \leq 8/5A_7S$$

$$\dot{D}_n = {}_0\dot{D} \text{ when } nh > 8/5A_7S \text{ and } ({}_0D - D_n)({}_0\dot{D}) > 0$$

$$\dot{D}_n = 0 \text{ otherwise.} \tag{26}$$

$$D_n = D_{n-1} + \frac{h}{2}(\dot{D}_n + \dot{D}_{n-1})$$

d. Case 4

$$\begin{aligned}\tau(\sin {}_0D - \sin D_0) = &\left(\frac{R}{\dot{D}_0} + \frac{R}{{}_0\dot{D}}\right)\cos\left(D_0 + \frac{8\dot{D}_0}{5A_7S}\right) \\ &+ \frac{1}{{}_0\dot{D}}\left[({}_0D - D_0) + \frac{8({}_0D - D_0)}{5A_7S}\right]\sin {}_0D \\ &- \frac{R}{\dot{D}_0}\cos D_0 + \frac{R}{{}_0\dot{D}}\cos {}_0D\end{aligned} \tag{27}$$

where $R = 180/\pi$ and ${}_0\dot{D}$ and $\dot{D}_0$ are in degrees.

When $\dot{D}_0$ and D_0 are zero

$$\tau = \frac{8}{5A_7S} + \frac{{}_0D}{{}_0\dot{D}} - \frac{R(1 - \cos {}_0D)}{{}_0\dot{D}\sin {}_0D} \tag{28}$$

In either instance,

$$D_n = D_0 \text{ when } nh \leq \tau$$

$$D_n = {}_0D \text{ when } nh > \tau \tag{29}$$

Restrictions

The validity of certain approximations made in deriving the formulas for cases 1 and 2 depends upon the speed of the submarine. The speed must satisfy the following inequality:

$$\frac{5}{8}A_7Sh < 1 \tag{30}$$

The value of A_7 for Submarines II and III was seen to be inaccurate by several orders of magnitude. For Submarine I, inequality 30 becomes:

$$hS < 64 \text{ yards} = 114 \text{ knot-seconds} \tag{31}$$

Limits on the dive angle are inherent in the equation for case 1. When $|D_{n-1}| > \frac{A_{11}}{A_9} S^2 \delta_{s_{n-1}}$, $|D_n - D_{n-1}|$ will decrease until $D_n - D_{n-1}$ changes sign and causes $|D_{n-1}|$ to become smaller than $A_{11} S^2 \delta_{s_{n-1}} / A_9$. This limit is not included in the formulations for cases 2 and 3. When the stern plane formulation is used in case 2, D_{n-1} can be used instead of ${}_0D$ when D_n exceeds $\frac{A_{11}}{A_9} S^2 \delta_{s_{n-1}}$. In the ${}_0\dot{D}$ formulation of case 2, and in case 3, D_n must be tested by the following formula:

$$|D_{n-1}| < |5A_7 S_0 \dot{D} / 8A_9| \qquad (32)$$

(derived from the above inequality for D_{n-1} using equation A-67)

When $|D_{n-1}|$ exceeds this value, ${}_0\dot{D}$ must be changed to prevent D_n from increasing further.

In cases where ${}_0\dot{D}$ is part of the input, the maximum allowable value which may be used for ${}_0\dot{D}$ depends upon the maximum allowable value of δ_s, as follows:

$$|{}_0\dot{D}| \max = (8A_{11} S / 5A_7) \delta_{s\,max} \qquad (33)$$

B.1.3 Formula Cross-References, Appendices B to A

Appendix B	Appendix A	Appendix B	Appendix A
1	27	18	50
2	37	19	25
3	43	20	25 ff.
4	25	21	12 ff.
5		22	74
6		23	67
7	15	24	72
8	8	25	73
9	7	26	77
10	26	27	81
11	13	28	83
12	28	29	84
13	22	30	63 ff.
14	36	31	63 ff.
15	49	32	
16	42	33	68
17	51		

B.2 SURFACE VESSEL

B.2.1 Symbols and Units

Symbol	Definition	Units	Positive Direction
θ	ship's heading	radians	clockwise from north
y	$\dot{\theta}$	radians/second	
α	side-slip angle	radians	clockwise from true course angle to heading
C	true course angle ($\theta - \alpha$)	radians	clockwise from north
$\dot{C}$	$y - \dot{\alpha}$	radians/second	
δ	rudder angle	radians	trailing edge to the left
V	velocity	feet/second	forward
a_1	(constant parameters of the surface vessel's handling characteristics)	1/feet^2	
a_2		1/feet^2	
a_3		1/feet	
a_4		1/feet	
a_5		1/feet	
a_6		none	
a_7		feet/radians^2	
a_8		1/feet	

Symbol	Definition
V_o	velocity for which surface vessel's engines are set
${}_o\dot{C}$	ordered course rate of change or steady-state rate of change of course in a turn
$\underline{\delta}$	rudder angle corresponding to ${}_o\dot{C}$
T	time required to move rudder from zero to $\underline{\delta}$
V_f	steady-state velocity in a turn

B. 2. 2 Kinematic Equations of a Surface Vessel's Motions

B. 2. 2. 1 Acceleration/Deceleration

Instructions

In the literature in which the dynamic equations for surface vessel motion were found, V_o is used to denote ordered speed, the speed for which the ship's engines are set. Therefore V_1 will be used to denote the initial speed of the vehicle at time $t = 0$. This is used in cases 2, 3, and 4. In all cases, the initial value of V is used for the first appearance of V_{n-1}.

Formulas

a. Cases 1 and 2

$$V_n - V_{n-1} = ha_8(V_o^2 - V_{n-1}^2) \tag{34}$$

b. Case 3

$$V_n - V_{n-1} = \frac{ha_8}{2} (V_o - V_1) \Bigg/ \left[\frac{V_o + 3V_1}{2(V_o + V_1)^2} \right]$$

$$\text{when } (V_o - V_n)(V_o - V_1) < 0$$

$$V_n = V_o \qquad \text{otherwise} \tag{35}$$

where V_1 is the original velocity at the start of the acceleration/deceleration.

c. Case 4

$$V_n = V_1 \qquad \text{when } nh < \frac{1}{2a_8}\left[\frac{V_o + 3V_1}{(V_o + V_1)^2}\right]$$

$$V_n = V_o \qquad \text{when } nh \geq \frac{1}{2a_8}\left[\frac{V_o + 3V_1}{(V_o + V_1)^2}\right] \tag{36}$$

Restrictions

For the above equations to be valid, V must satisfy the following inequality, where V is the maximum speed encountered during the maneuver.

$$0 < Vh < \frac{1}{2a_8} \tag{37}$$

As concerns the three vehicles tested, this becomes

For Surface Vessel I , $0 < Vh < 1{,}014$ feet = 600 knot-seconds
For Surface Vessel II , $0 < Vh < 1{,}075$ feet = 637 knot-seconds
For Surface Vessel III, $0 < Vh < 2{,}222$ feet = 1,316 knot-seconds (38)

B.2.2.2 Turn

Instructions

All four cases give $\dot{C}_n$ as output. C is updated, using the formula:

$$C_n = C_{n-1} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1}) \tag{39}$$

Case 1 has δ_n as input at each iteration $\delta = r_1 + r_2 t$. While the rudder is moving, $r_2 =$ (quired in addition. Initialization requires V_o, the speed for which the engines are set. When entering a turn from a straight-course situation, V_o is also the initial value of the speed. Initial V_{n-1}, y_{n-1}, α_{n-1}, $\dot{C}_{n-1}$, and C_{n-1} are all set from the values for these quantities when the maneuver begins.

Case 2 is initialized in the same way. However, instead of updating δ_n at each iteration, only the principal value ($\underline{\delta}$) of δ_n is used. $\underline{\delta}$ may either be a direct input or computed using the relationship

$$\underline{\delta}\xi V_f = {}_o\dot{C} \tag{40}$$

when the input is in the form of an ordered turn rate.

In cases 3 and 4, ${}_o\dot{C}$ is part of the initialization input, but $\underline{\delta}$ can be used with equation 40 .

The two formulas for V_f use $\underline{\delta}$ and ${}_o\dot{C}$ respectively.

Formulas

$$\xi = \frac{a_1a_5 - a_2a_4}{a_3a_4 - a_1a_6} \tag{41}$$

$$\eta = \frac{a_2a_6 - a_3a_5}{a_3a_4 - a_1a_6} \tag{42}$$

$$V_f = V_o \Big/ \sqrt{1 - a_7a_6{}^2\xi^2\underline{\delta}^2/a_8} \tag{43}$$

$$V_f = \sqrt{V_o{}^2 + a_6{}^2\,{}_o\dot{C}^2a_7/a_8} \tag{44}$$

$${}_o\dot{C} = \underline{\delta}\xi V_f$$

a. Case 1

$$y_n - y_{n-1} = \frac{V_{n-1}h}{a_4}\left[(y_{n-1} - V_{n-1}\xi\delta_{n-1})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right] \tag{45}$$

$$\dot{\alpha}_n - \dot{\alpha}_{n-1} = \frac{V_{n-1}ha_6}{a_4}\left[(y_{n-1} - V_{n-1}\xi\delta_{n-1})(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right]$$

$$+ V_{n-1}h(a_4\dot{\alpha}_{n-1} + a_5r_2) \tag{46}$$

where $r_2 = (\delta_n - \delta_{n-1})/h$

$$V_n - V_{n-1} = h\left[a_8(V_o^2 - V_{n-1}^2) + a_7(\dot{\alpha}_{n-1} - a_6y_{n-1})^2\right] \tag{47}$$

$$\dot{C}_n = y \ \ - \dot{\alpha}_n \tag{48}$$

$$C_n = C_{n-1} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1})$$

b. Case 2

$y_n = 0$ when $nh < T$

$$y_n - y_{n-1} = \frac{V_{n-1}h}{a_4}\left[\left(y_{n-1} - \frac{{}_o\dot{C}V_{n-1}}{V_f}\right)(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right] \tag{49}$$

when $nh \geq T$

$\dot{\alpha}_n = 0$ when $nh < T$

$$\dot{\alpha}_n - \dot{\alpha}_{n-1} = \frac{V_{n-1}ha_6}{a_4}\left[\left(y_{n-1} - \frac{{}_o\dot{C}V_{n-1}}{V_f}\right)(a_3a_4 - a_1a_6) + a_1\dot{\alpha}_{n-1}\right]$$

$$+ Vha_4\dot{\alpha}_{n-1}$$

when $nh \geq T$ (50)

$$V_n - V_{n-1} = h\left[a_8(V_o^2 - V_{n-1}^2) + a_7(\dot{\alpha}_{n-1} - a_6y_{n-1})^2\right] \tag{51}$$

$$\dot{C}_n = y_n - \dot{\alpha}_n$$

$$C_n = C_{n-1} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1})$$

c. **Case 3**

$$\dot{C}_n = \dot{C}_{n-1} + h({}_o\dot{C} - \dot{C}_o)\left[T - \frac{2}{V_f\xi}\left(\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right)\right]^{-1}$$

when $(\dot{C}_o - {}_o\dot{C})\dot{C}_n > (\dot{C}_o - {}_o\dot{C})_o\dot{C}$

$\dot{C}_n = {}_o\dot{C}$ otherwise **(52)**

$$V_n = V_{n-1} + a_8h(V_f - V_o)\left[\frac{V_f + 3V_o}{(V_f + V_o)^2}\right]^{-1}$$

when $(V_1 - V_f)V_n > (V_1 - V_f)V_f$

$V_n = V_f$ otherwise. **(53)**

$$C_n = C_{n-1} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1})$$

d. **Case 4**

$\dot{C}_n = \dot{C}_o$ when $nh < \frac{T}{2} - \frac{1}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$

$\dot{C}_n = {}_o\dot{C}$ when $nh \geq \frac{T}{2} - \frac{1}{\xi V_f}\left[\frac{a_4\xi - a_1\eta}{a_3a_4 - a_1a_6} - \eta\right]$ **(54)**

$V_n = V_o$ when $nh < \frac{1}{2a_8}\left[\frac{V_f + 3V_o}{(V_f + V_o)^2}\right]$

$V_n = V_f$ when $nh \geq \frac{1}{2a_8}\left[\frac{V_f + 3V_o}{(V_f + V_o)^2}\right]$ **(55)**

$$C_n = C_{n-1} + \frac{h}{2}(\dot{C}_n + \dot{C}_{n-1})$$

Restrictions

The iteration interval is restricted in size by the inequality

$[(a_3 + a_4) - \sqrt{(a_3 + a_4)^2 - 4(a_3a_4 - a_1a_6)}\,]\frac{Vh}{2} < 1$ where V is the largest velocity encountered in the turn maneuver. For the three surface vessels for which response curves are displayed in Appendix C, this becomes

For Surface Vessel I , $0 < Vh < 86.5$ feet = 51.2 knot-seconds
For Surface Vessel II , $0 < Vh < 71.8$ feet = 42.5 knot-seconds
For Surface Vessel III, $0 < Vh < 185$ feet = 110 knot-seconds

B.2.3 Formula Cross-References, Appendices B to A

Appendix B	Appendix A	Appendix B	Appendix A
34	137	45	114
35	147	46	120
36	150	47	137
37	135	48	
38	136	49	122
39		50	123
40	102	51	137
41	97	52	126
42	98	53	147
43	140	54	127
44	139	55	150

B.3 POSITION UPDATING

B.3.1 Orientation of Vessels

Instructions

The following equations represent the orientation of either submarine or surface vessel ($D = 0$ for surface vessel).

$$\dot{X} = S \cos D \sin C$$

$$\dot{Y} = S \cos D \cos C$$

$$\dot{Z} = -S \sin D \tag{56}$$

Updating formulas are given for X and Y with $D = 0$. The formula for X can be used for Z if multiplied by -1. All values of C, $\dot{C}$ and S are obtained from the appropriate formulas in sections B.1 and B.2.

Formulas

$$X_n - X_{n-1} = (S_n + S_{n-1})\frac{h}{2}[\sin C_{n-1} + (\dot{C}_n + \dot{C}_{n-1})\frac{h}{4} \cos C_{n-1}] \tag{57}$$

$$Y_n - Y_{n-1} = (S_n + S_{n-1})\frac{h}{2}[\cos C_{n-1} - (\dot{C}_n + \dot{C}_{n-1})\frac{h}{4} \sin C_{n-1}] \tag{58}$$

where

$$\sin C_{n-1} = [1 - (\dot{C}_{n-1} + \dot{C}_{n-2})^2 \frac{h^2}{8}]\sin C_{n-2} + (\dot{C}_{n-1} + \dot{C}_{n-2})\frac{h}{2} \cos C_{n-2} \tag{59}$$

$$\cos C_{n-1} = [1 - (\dot{C}_{n-1} + \dot{C}_{n-2})^2 \frac{h^2}{8}] \cos C_{n-2} - (\dot{C}_{n-1} + \dot{C}_{n-2}) \frac{h}{2} \sin C_{n-2} \qquad (60)$$

where $\dot{C}$ must always be expressed in radians.

Restrictions

The only restriction is on the size of $\dot{C}h$. Accuracy will be very good if $\dot{C}h \leq 10°$. It will be fair if $10° < \dot{C}h \leq 20°$. $\dot{C}h > 20$ will lose much accuracy as n increases. $\dot{C}h > 40°$ will be very inaccurate, and $\dot{C}h > 57°$ completely wrong.

B.3.2 Formula Cross-References, Appendices B to A

Appendix B	Appendix A	Appendix B	Appendix A
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57	86	60	87
58	86		

B.4 AIRCRAFT

B.4.1 Symbols and Units

Symbol	Definition	Units	Positive Direction
v	velocity or airspeed	knots	forward
m	mass of the aircraft	pound-seconds/knots	
h_o	altitude	1000 feet	up
ω	course rate of change in a coordinated turn	radians/seconds	clockwise
ϕ	bank angle in a coordinated turn	degrees	right wing down
p	roll rate	degrees/second	right wing moving down
$(\Delta C)_f$	course change while rolling out of a coordinated turn	radians	
g	acceleration due to gravity	knots/second	
$_o\phi$	ordered bank angle		
θ	altitude or pitch angle	radians	up from horizontal plane
d	angle of attack	radians	up from horizontal plane
γ	direction of flight in the veritical plane	radians	up from horizontal plane
b	(constant parameters	knots	
c	of the aircraft's	knots x pounds	
k	handling	$knots^2$	
A	characteristics)	knots	

Symbol	Definition
v_m	velocity the aircraft would have, at a given thrust and altitude, if $\gamma = 0$
${}_o v$	ordered velocity in acceleration/deceleration

B.4.2 Kinematic Equations of an Aircraft's Motions

B.4.2.1 Acceleration/Deceleration

Instructions

${}_o v$ is ordered speed, the speed for which the aircraft's engines are set; m is the mass of the aircraft. The first value of v_{n-1} is the value of v before the acceleration takes place.

Formulas

$$v_n - v_{n-1} = ch({}_o v - v_{n-1})/m({}_o v - b)(b - v_{n-1}) \tag{61}$$

For the F-100A,

$$b = 711.636\ (1-0.00551h_o) \tag{62}$$

$$c = 26h_o - 1000 \tag{63}$$

Restrictions

$v \leq b, \ {}_o v \leq b$

B.4.2.2 Turn

Instructions

The aircraft rolls into the turn with roll rate p_i and out of the turn with roll rate p_o.

If the aircraft has been ordered to a specific course change $_oC$, then it starts to roll out of the turn before it reaches this course change. The point where it starts to roll out is expressed by $_oC - (\Delta C)_f$, where $(\Delta C)_f$ is the course change while the aircraft is rolling out of the turn. The formula for $(\Delta C)_f$ is shown in equation 66, where p_o is in radians per second. Figure D-1 in Appendix D shows how these equations are used to construct a model to simulate a coordinated turn.

$$C_n - C_{n-1} = \frac{hg}{v} \tan \phi \tag{64}$$

$$\phi_n = \phi_{n-1} + p_i \quad \text{while} \left| \phi_n \right| < {_o\phi} \tag{65}$$

$$(\Delta C)_f = \frac{g}{2vp_o} \log(1 + \tan^2 {_o\phi}) \tag{66}$$

$$\phi_n = {_o\phi} \quad \text{until} \quad C = {_oC} - (\Delta C)_f \tag{67}$$

$$T = \text{time at which} \quad C = {_oC} - (\Delta C)_f \tag{68}$$

$$\phi_n = \phi_{n-1} - p_o(t - T) \quad \text{until} \quad \phi_n = 0 \tag{69}$$

Restrictions

$h < {_o\phi}/3p$

B.4.2.3 Climb

Instructions

The equations for climb relate the variables of the aircraft that are affected by the climb. v_m is the air speed the aircraft would have if it were proceeding at level flight with the same throttle setting. θ is the attitude or pitch angle of the aircraft. A is the slope of the graph of sin θ versus v. It is not constant but varies somewhat with altitude, so an average value is used. γ is the angle between the horizontal and the actual direction of fligh

Equations 72 and 73 are simpler versions of equations 70 and 71.

An ordered climb is achieved by changing θ at an ordered rate until the proper climb rate is attained. The rate of change in θ is accomplished by ordering the vehicle to turn upward or downward so as to increase or decrease by a given amount the number of "g's" the

pilot and the aircraft will experience. If the number of g's change is given by G, then ω, the rate of change of θ, is expressed as $\omega = 1091\frac{G}{v}$ when v is in knots and ω in degrees per second.

The various equations used to describe a climb must be combined with tests on γ and v and arranged in such a way so that a climb will actually take place. For instance, at the start of a climb a certain value of γ is determined according to the speed of the aircraft, etc. Call it γ_0, then θ is incremented by $h\omega$ at each iteration and updated using equation 71 or 73. Then γ is tested to see if it is greater than or equal to ${}_0\gamma$. If not, θ is incremented again. If it is, then γ is set equal to ${}_0\gamma$ and G set equal to zero. A similar process is repeated whenever ${}_0\gamma$ must be changed.

A flow chart (Figure D-2) is included in Appendix D to show how this is done. It is set up so that the aircraft will either climb to an ordered altitude and level off or will attain an ordered heading angle (${}_0\gamma$) and maintain it. Speed change to ${}_0v$ is included in this model for completeness. Values for G of +0.5 and -0.5 were chosen arbitrarily. G can be any value consistent with the structural limitations of aircraft and pilot.

Formulas

$$v = v_m - A(\sin\theta - \frac{k}{v_m^2}) \qquad (70)$$

$$\sin\gamma = \frac{v^2 \sin\theta - k}{v^2 - k\sin\theta} \qquad (71)$$

or

$$v = v_m - A\sin\gamma \qquad (72)$$

$$\sin\gamma = \sin\theta - \frac{k}{v^2} \qquad (73)$$

Restrictions

$\alpha < 10°$ where $\alpha = \theta - \gamma$

B.4.3 Formula Cross-References, Appendices B to A

Appendix B	Appendix A	Appendix B	Appendix A
61	156	68	
62	157	69	
63	158	70	170
64	159	71	171
65		72	172
66	160	73	173
67			

APPENDIX C

RESPONSE CURVES

Appendix C (CONFIDENTIAL) is issued separately as Supplement 1, Technical Report: NAVTRADEVCEN 1407-2

APPENDIX D

FLOW CHARTS

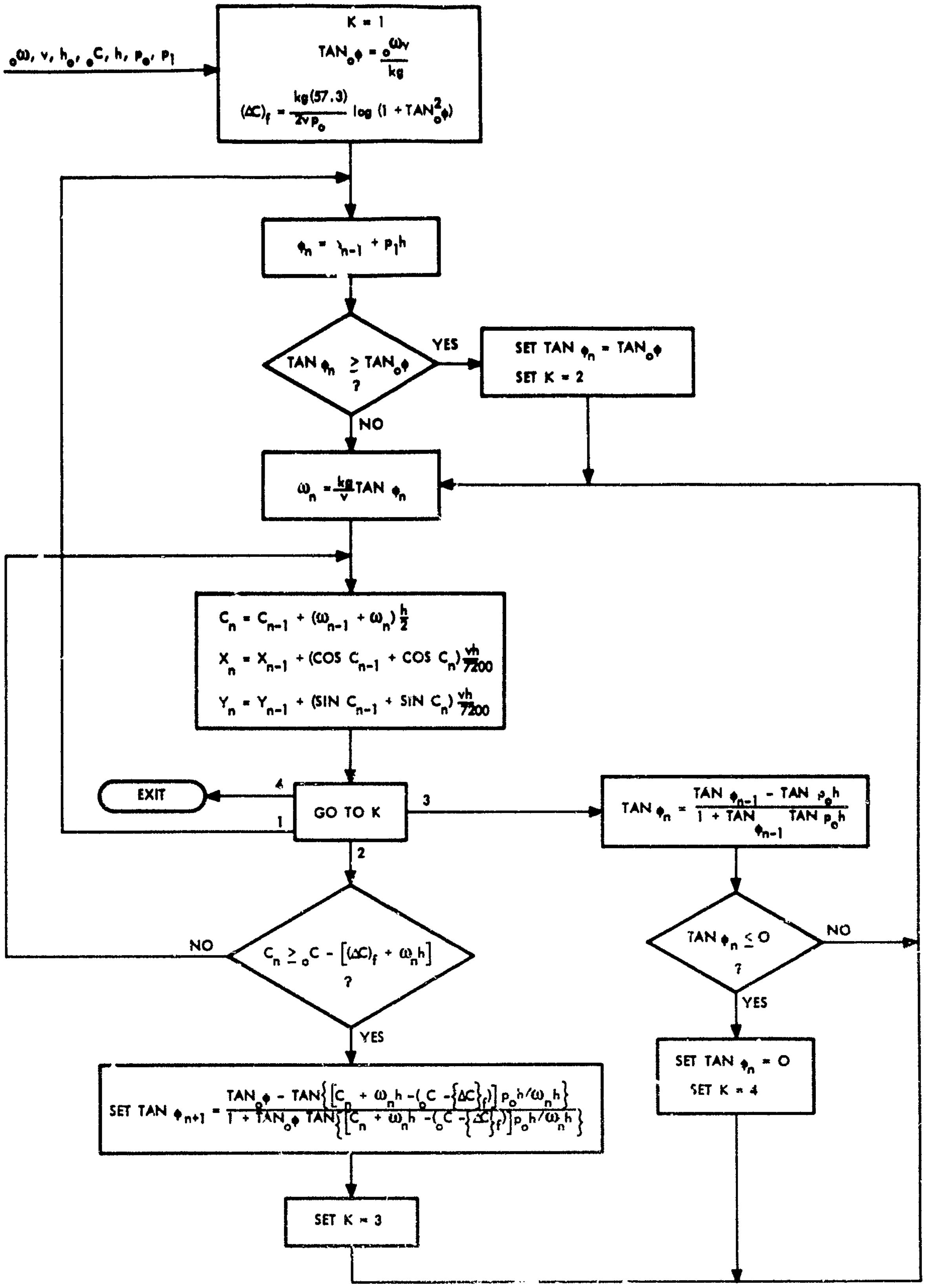
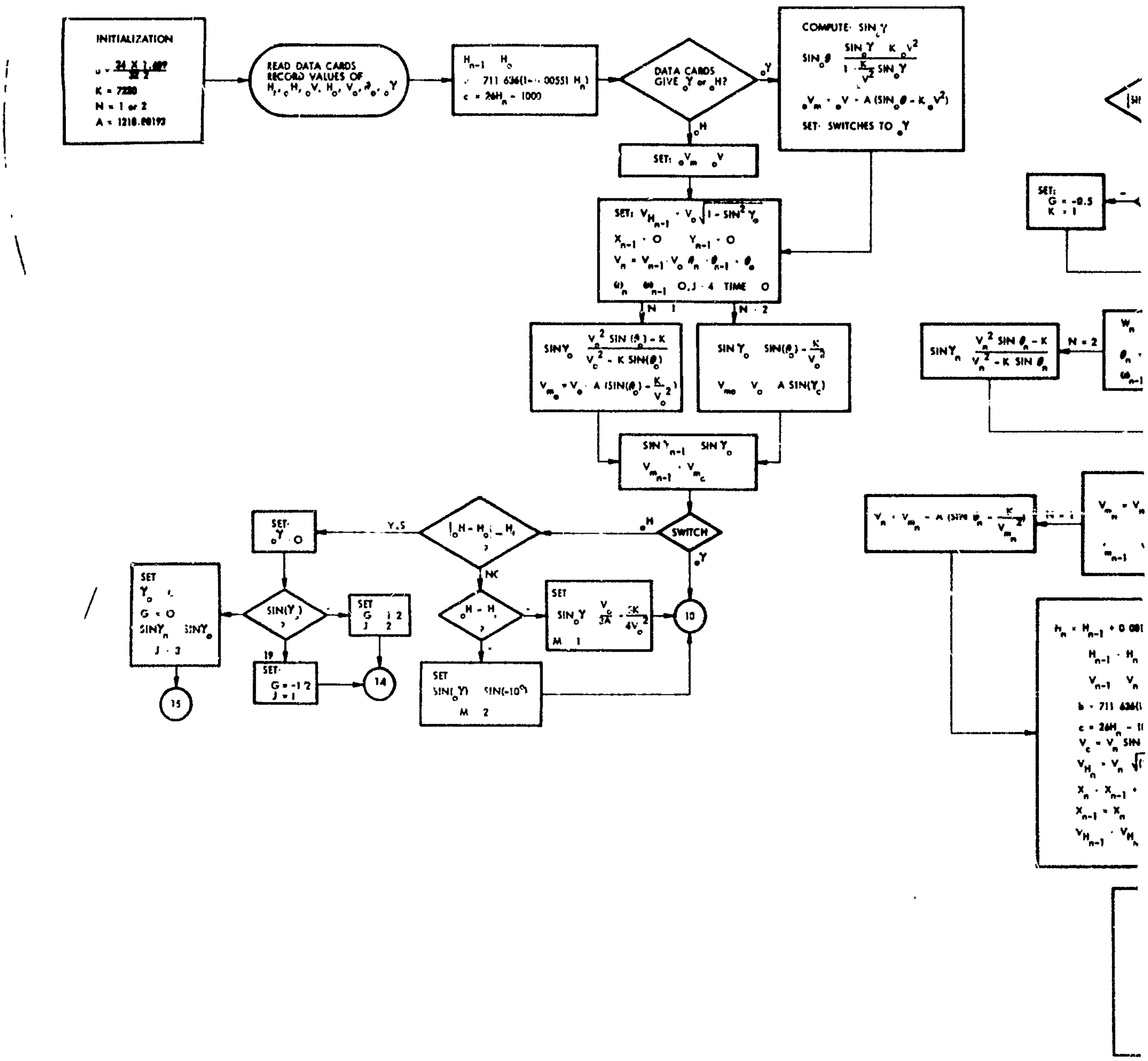


Figure D-1. Flow Chart, Aircraft Turn



INITIALIZATION
K = 7200
N = 1 or 2
A = 1210.88192
READ DATA CARDS RECORD VALUES OF
DATA CARDS GIVE Y or H?
COMPUTE SIN Y
SET SWITCHES TO Y
SET
SWITCH
SET
SET
SET
SET
SET
SET
SET
SET
SET

A

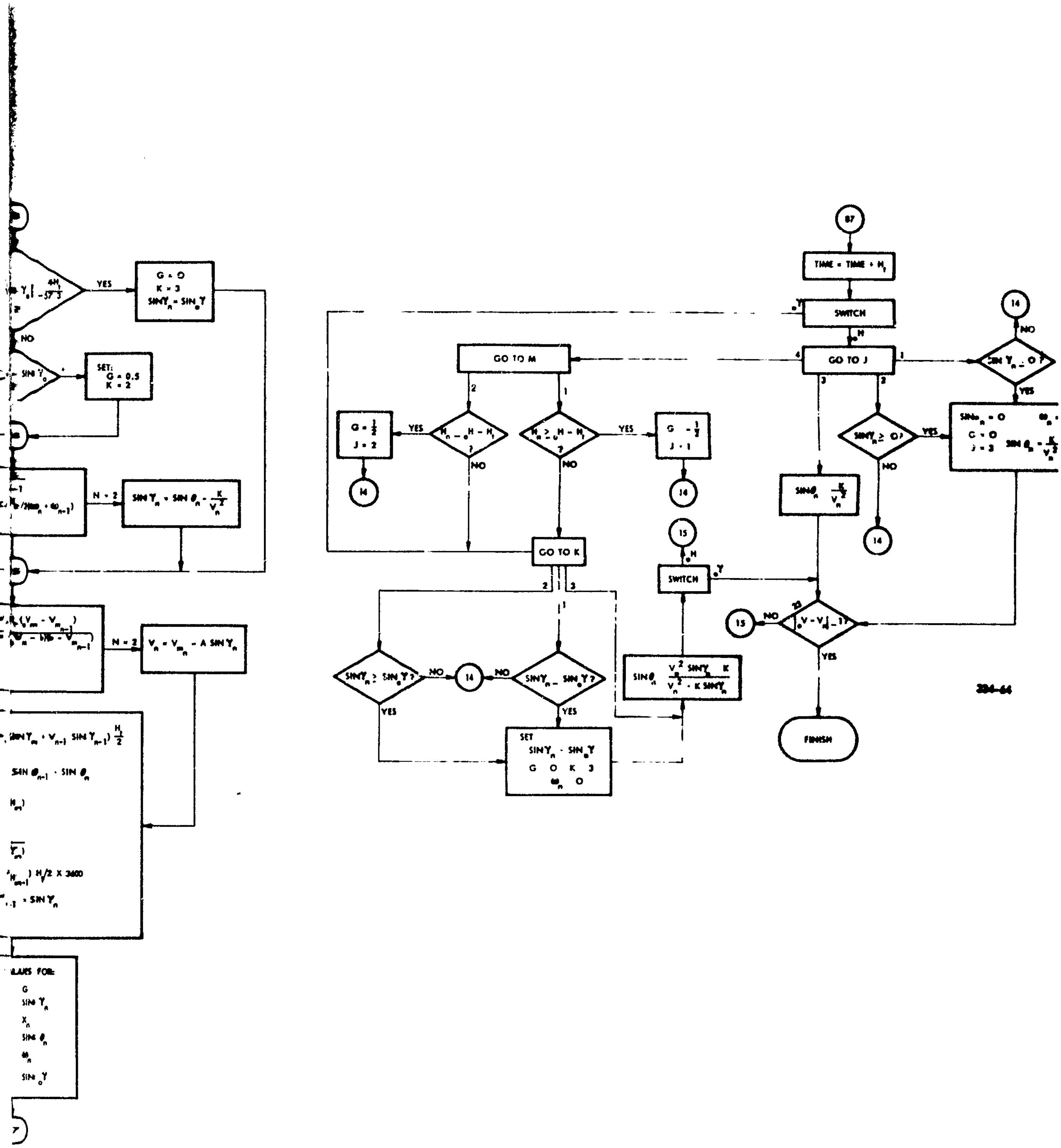


Figure D-2. Flow Chart, Aircraft Climb

B

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13. ABSTRACT This is a report on the development and use of kinematic vehicle simulatic equations. It provides the appropriate kinematic equations for vehicle simulation systems and the criteria for their selection.

The kinematic equations in this report are difference equations involving position, speed and acceleration. Difference in position is a function of velocity and difference in velocity is a function of acceleration. Acceleration algorithms, necessary for any system of kinematic equations, are incorporated directly into the difference equations for velocity. They are based on the parameters of the maneuve being executed and the handling characteristics of the vehicle being simulated. Equations have been developed for simulating maneuvers of the submarine, surface vessel and aircraft. The kinematic equations describing the motion of submarine and surface vessels cover four broad simulation categories: operator, command, instructor and program controlled maneuvers. Operator controlled maneuvers are those in which control surface deflections are used to effect the maneuvers. These maneuvers follow the actual motion of the vehicle very closely. Command controllec maneuvers are effected, by ordered rates of change such as ordered course rate of change or ordered dive rate of change or by an ordered speed. The vehicle changes to ordered rate of change in a characteristic manner. Instructor controlled maneu are simulated by using a constant change in course rate of change, dive rate or speed until the ordered value is achieved. Program controlled maneuvers are simulated by an abrupt change in these variables after an appropriate time delay. Both these maneuvers keep the position of the vehicle very well. The iteration intervals used are one second or greater. The maximum iteration intervals and the computer storage and timing demands of each model are presented.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Kinematic Equations Difference Equations Ordered Maneuvers Digital Computation Simulated Motion						

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